Conformal mapping :-

A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if $f'(z_0) \neq 0$.

The point at which the mapping $w = f(z)$ is not conformal.

i.e.,) $f'(z) = 0$ is called a critical point of the mapping.

If the transformation $w = f(z)$ is conformal at a point the inverse transformation $z = f^{-1}(w)$ is also conformal at the corresponding point.

Some standard transformations :-

1. Translation :-

The transformation $w = c + z$, where c is a complex constant, represents a translation.

Let $z = x + iy$

$w = u + iv$

$c = a + ib$

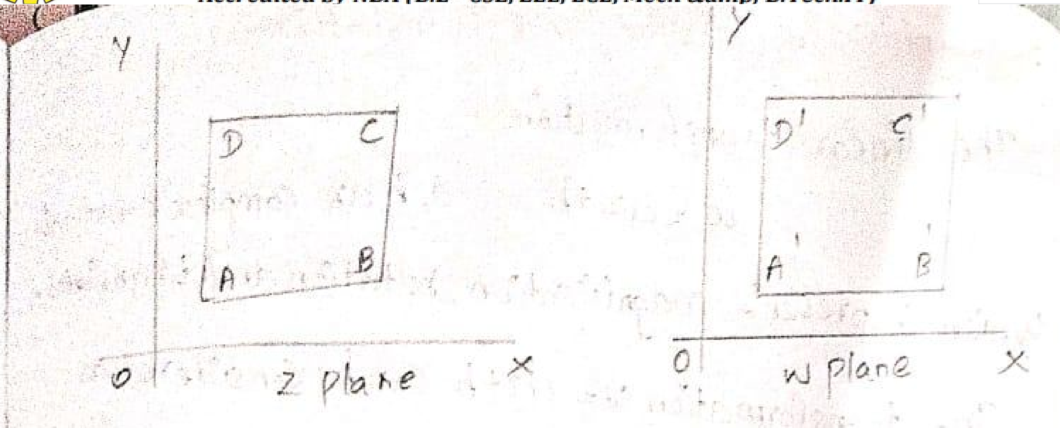
Given $w = z + c$

$u + iv = x + iy + a + ib$

$u + iv = (x + a) + i(y + b)$

Equating the real and imaginary parts, we get

$u = x + a, \quad v = y + b$



2. Magnification:-

The transformation $w = cz$, where c is a real constant, represents magnification.

$$u + iv = c(x + iy)$$

$$u + iv = (cx + icy)$$

$$u = cx, \quad v = cy$$

The image of the point (x, y) is the point (cx, cy).

3. Magnification and Rotation:-

The transformation $w = cz$, where c is a complex constant represents both magnification and rotation.

Let $z = re^{i\theta}$

$$w = Re^{i\phi} \quad \text{and} \quad c = ae^{i\alpha}$$

$$Re^{i\phi} = (ae^{i\alpha})(re^{i\theta}) = are^{i(\alpha+\theta)}$$

The transformation equations are $R = ar$, $\phi = \theta + \alpha$.

The point (r, θ) in z-plane is mapped on to the point $(ar, \theta + \alpha)$.

4. Magnification, Rotation and Translation:

The linear transformation

$w = az+b$ a, b are complex constants.

which represents magnification, rotation and translation.

The transformation $w = az+b$ can be considered as combination of two simple transformations.

$w_1 = az$ and $w = w_1+b$

$w_1 = az$ represents magnification by $|a|$ and rotation through amp (a).

$w = w_1+b$ represents translation by a vector representing b.

5. Inversion and Reflection:

The transformation $w = \frac{1}{z}$ represents inversion with respect to the unit circle $|z|=1$, followed by reflection in the real axis.

$$w = \frac{1}{z} \quad (\text{or}) \quad z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2} \rightarrow (2)$$

$\hookrightarrow (1)$

The general equation of circle in z plane is

$$x^2+y^2+z^2+2gx+2fy+c=0 \rightarrow (3)$$

sub (1), (2) in (3) equ, we get

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+2g\left(\frac{u}{u^2+v^2}\right)+2f\left(\frac{-v}{u^2+v^2}\right)+c=0$$

$$c(u^2+v^2)+2gu-2fv+1=0 \rightarrow (4)$$

which is the equation of circle in w plane.

Problems:

1) what is the region of the w-plane into which the rectangular region in the Z-plane bounded by the lines $x=0, y=0, x=1, y=2$ is mapped under the transformations $w=z+(2-i)$.

Sol: Given $w=z+(2-i)$

ie, $u+iv = x+iy+(2-i) = (x+2)+i(y-1)$.

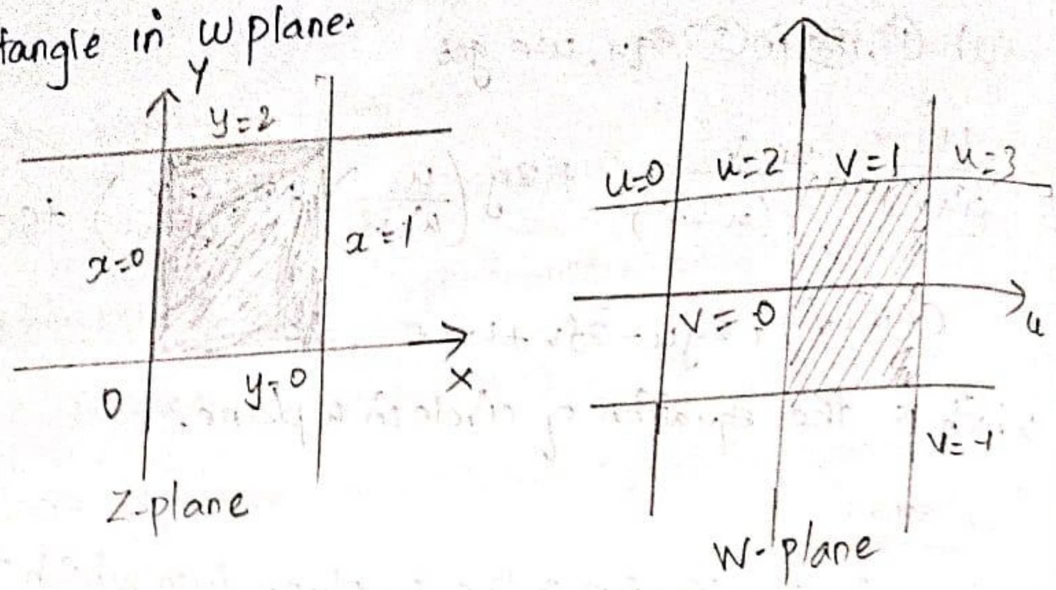
Equating the real and imaginary parts

$$u=x+2, \quad v=y-1.$$

Given boundary lines are the transformal boundary lines are

$x=0$	$u=0+2=2$
$y=0$	$v=0-1=-1$
$x=1$	$u=1+2=3$
$y=2$	$v=2-1=1$

Hence the lines $x=0, y=0, x=1, y=2$ are mapped into the lines $u=2, v=-1, u=3, v=1$ which form a rectangle in w plane.



2) Find the image of the circle $|z|=1$ by the transformation $w = z+2+4i$.

Sol:. Given $w = z+2+4i$

$$u+iv = x+iy+2+4i$$
$$= (x+2)+i(y+4)$$

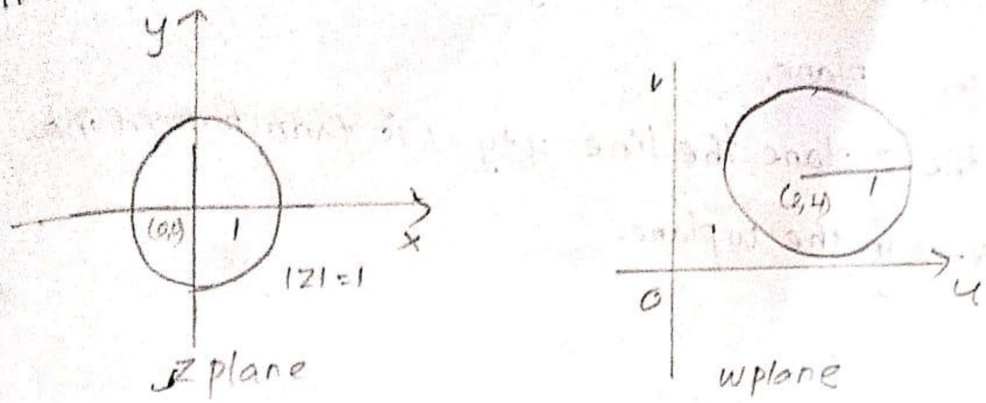
Equating the real and imaginary parts,

$$u = x+2, \quad v = y+4$$
$$x = u-2, \quad y = v-4$$

Given :. $|z|=1$

$$x^2+y^2=1$$
$$(u-2)^2+(v-4)^2=1$$

Hence the circle $x^2+y^2=1$ is mapped into $(u-2)^2+(v-4)^2=1$ in w plane which is also a circle with centre $(2,4)$ and radius 1.



3) Determine the region 'D' of the w-plane into which the triangular region D enclosed by the lines $x=0, y=0$, $x+y=1$ is transformed under the transformation $w=2z$.

Sol:- Let $w=u+iv$

$z=x+iy$

Given, $w=2z \Rightarrow u+iv=2(x+iy)$

$u+iv=2x+i2y$

$u=2x, \; v=2y$

Given region (D) whose boundary lines are

$x=0$

$y=0$

$x+y=1$

Transformed region D whose boundary lines are

$u=0$

$v=0$

$\frac{u}{2}+\frac{v}{2}=1$

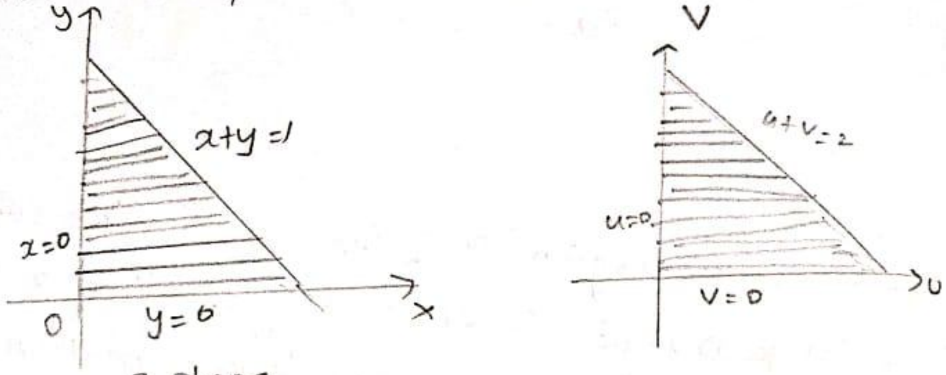
ie., $u+v=2$

$\left(x=\frac{u}{2}, \; y=\frac{v}{2}\right)$

In the z-plane the line $x=0$ is transformed into $u=0$ in the w plane.

In the z-plane the line $y=0$ is transformed into $v=0$ in the w plane.

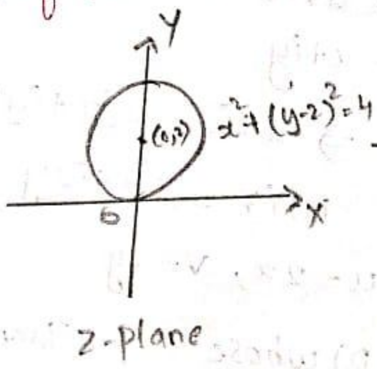
In the z-plane the line $x+y=1$ is transformed into $u+v=2$ in the w plane.

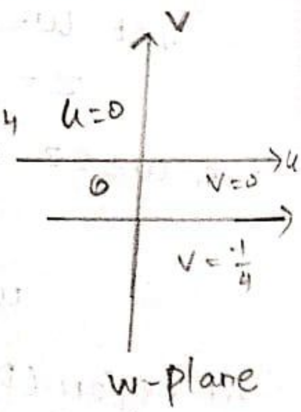


z-plane

4) Find the image of $|z-2i|=2$ under the transformation $w=\frac{1}{z}$.

Sol. Given $w=\frac{1}{z}$





i.e., $z=\frac{1}{w}$

$w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

$$x=\frac{u}{u^2+v^2} \rightarrow (1), \quad y=\frac{-v}{u^2+v^2} \rightarrow (2)$$

Given: $|z-2i| = 2$

$$|x+iy-2i| = 2$$

$$|x+(y-2)i| = 2$$

$$x^2+(y-2)^2 = 4$$

$$x^2+y^2+4-4y = 4$$

$$x^2+y^2-4y = 0 \longrightarrow (3)$$

sub (1), (2) in equ (3), we get

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0$$

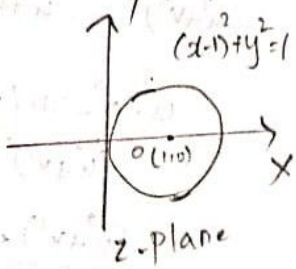
$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$\frac{(u^2+v^2)(1+4v)}{(u^2+v^2)^2} = 0 \qquad (u^2+v^2 \neq 0)$$

$$1+4v = 0$$

which is a straight line in w plane.

5) Find the image of the circle $|z-1| = 1$ in the complex plane under the mapping $w = \frac{1}{z}$.



Sol:- The given transformation $w = \frac{1}{z}$

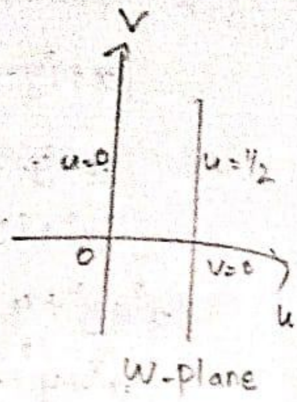
ie,) $z = \frac{1}{w}$

Let $w = u+iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$= \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$



Equating the real and imaginary Parts,

$$x = \frac{u}{u^2+v^2} \rightarrow (1) \qquad y = \frac{-v}{u^2+v^2} \rightarrow (2)$$

Given: $|z-1| = 1$

$$|x+iy-1| = 1$$

$$|(x-1)+iy| = 1$$

$$(x-1)^2+y^2 = 1$$

$$x^2-2x+1+y^2 = 1$$

$$x^2-2x+y^2 = 0 \longrightarrow (3)$$

Sub (1) & (2) in (3) equ, we get

$$\left(\frac{u}{u^2+v^2}\right)^2 - 2\left(\frac{u}{u^2+v^2}\right) + \left(\frac{-v}{u^2+v^2}\right)^2 = 0$$

$$\frac{u^2}{(u^2+v^2)^2} - \frac{2u}{u^2+v^2} + \frac{v^2}{(u^2+v^2)^2} = 0$$

$u^2+v^2 \neq 0$

$$\frac{u^2+v^2-2u(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$\frac{(u^2+v^2)(1-2u)}{(u^2+v^2)^2} = 0$$

$$\frac{1-2u}{u^2+v^2} = 0$$

$$1-2u = 0$$

$$1 = 2u$$

$$\frac{1}{2} = u$$

which is a straight line in w-plane.