Unit-IV

Correlation and Spectral Densities

Auto Correlation:

Auto correlation of a random Process $\{X(t)\}$ is defined by $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

Properties:

1. The mean square value of the random process may be obtained from the auto correlation function $R_{XX}(\tau)$, by putting $\tau=0$.

ie., $E[X^2(t)] = R_{XX}(0)$

Proof:

WKT, $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

Put $\tau=0$,

$$R_{XX}(0) = E[X(t)X(t)]$$

$$R_{XX}(0) = E[X^2(t)]$$

2. $R_{XX}(\tau)$ is an even function of τ.

ie., $R_{XX}(\tau) = R_{XX}(-\tau)$

Proof:

WKT $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

Replace τ by $-\tau$

$$R_{XX}(-\tau) = E[X(t)X(t-\tau)]$$

Now $t-\tau = t_1 \Rightarrow t = \tau + t_1$

$$\therefore R_{XX}(-\tau) = E[X(\tau+t_1)X(t_1)]$$

$$= E[X(t_1)X(t_1+\tau)]$$

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

3]. The maximum value of $R_{XX}(\tau)$ is attained at the point $\tau = 0$

ie, $|R_{XX}(\tau)| \leq R_{XX}(0)$

4]. If the process $X(t)$ contains a periodic component then $R_{XX}(\tau)$ will also contain periodic component of the same period.

5]. $\lim\limits_{|\tau| \to \infty} R_{XX}(\tau) = \overline{X}^2$

Problems :

1]. Check whether the following functions are valid Autocorrelation functions.

i). $R_{XX}(\tau) = \cos\tau + \frac{|\tau|}{T}$ [Valid]

ii). $5\sin \pi\tau$ [Not valid]

iii) $\tau^3 + \tau^2$ [Not valid]

iv) $\frac{1}{1+9\tau^2}$ [Valid]

Soln.

i). $R_{XX}(-\tau) = \cos(-\tau) + \frac{|-\tau|}{T} = \cos\tau + \frac{|\tau|}{T}$

$= R_{XX}(\tau)$

2]. Determine the mean & variance of the process given that the auto correlation function

$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$

Soln.

By property, $R_{XX}(0) = E[X^2(t)]$

Now, $R_{XX}(0) = 25 + \frac{4}{1+0} = 29 = E[X^2(t)]$

and $\lim_{|\tau| \to \infty} R_{XX}(\tau) = \lim_{|\tau| \to \infty} \left[25 + \frac{4}{\infty}\right]$

$= 25 = E[X(t)]^2$

$\Rightarrow E[X(t)] = 5, \quad E[X^2(t)] = 29$

$Var[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2 = 29 - 25$

$Var[X(t)] = 4$

3]. A Stationary random process has an auto correlation function and is given by.

$R_{XX}(\tau) = \dfrac{25\tau^2 + 36}{6.25\tau^2 + 4}$. find the mean & variance

Soln. By Properties,

i). $\bar{X}^2 = \lim_{|\tau| \to \infty} R_{XX}(\tau)$

$= \lim_{|\tau| \to \infty} \dfrac{25\tau^2 + 36}{6.25\tau^2 + 4} = \lim_{|\tau| \to \infty} \dfrac{\tau^2\left[25 + \frac{36}{\tau^2}\right]}{\tau^2\left[6.25 + \frac{4}{\tau^2}\right]}$

$= \lim_{|\tau| \to \infty} \dfrac{25 + \frac{36}{\tau^2}}{6.25 + \frac{4}{\tau^2}}$

$= \dfrac{25}{6.25}$

$\bar{X}^2 = 4$

$\bar{X} = 2 = E[X(t)]$

ii). $E[X^2(t)] = R_{XX}(0)$

$= \dfrac{25(0) + 36}{6.25(0) + 4} = 9$

$\therefore Var[X(t)] = E[X^2(t)] - [E[X(t)]]^2 = 9 - 4 = 5$