Cross Spectral Density (or) Cross Power Spectral density

Consider two jointly WSS random processes $X(t)$ and $Y(t)$.

Let $R_{XY}(\tau)$ and $R_{YX}(\tau)$ be their cross correlation functions.

Then the cross spectral densities are,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

Properties:

1. $S_{xy}(\omega) = S_{yx}(-\omega)$

Proof:

Now, $$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{yx}(-\tau)\, e^{-i\omega\tau}\, d\tau \qquad \because R_{xy}(\tau) = R_{yx}(-\tau)$$

$$= \int_{\infty}^{-\infty} R_{yx}(\tau_1)\, e^{i\omega\tau_1}\, (-d\tau_1)$$

put $\tau_1 = -\tau$

when $\tau = \infty \Rightarrow \tau_1 = -\infty$

$\tau = -\infty \Rightarrow \tau_1 = \infty$

$$= \int_{-\infty}^{\infty} R_{yx}(\tau_1)\, e^{i\omega\tau_1}\, d\tau_1$$

$$= S_{yx}(-\omega)$$

$$\therefore S_{xy}(\omega) = S_{yx}(-\omega)$$

2. Real part of $S_{xy}(\omega)$ is an even function of ω. ie, $S_{xy}(\omega) = S_{xy}(-\omega)$

Proof:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau)\, (\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

Real part of $S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\cos\omega\tau\, d\tau$

$$Re\left[S_{xy}(-\omega)\right] = \int_{-\infty}^{\infty} R_{xy}(\tau)\cos(-\omega)\tau \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau)\cos\omega\tau \, d\tau$$

$$= Re\left[S_{xy}(\omega)\right] \quad \because \cos(-\theta) = \cos\theta$$

$\therefore Re\left[S_{xy}(\omega)\right]$ is an even function of ω.

3. Imaginary part of $S_{xy}(\omega)$ is an odd function of ω.

Proof :

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau)(\cos\omega\tau - i\sin\omega\tau) \, d\tau$$

$$Im\left[S_{xy}(\omega)\right] = -\int_{-\infty}^{\infty} R_{xy}(\tau)\sin\omega\tau \, d\tau \longrightarrow (1)$$

$$Im\left[S_{xy}(-\omega)\right] = -\int_{-\infty}^{\infty} R_{xy}(\tau)\sin(-\omega\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau)\sin(\omega\tau) \, d\tau \quad \left[\because \sin(-\theta) = -\sin\theta\right]$$

$$= -Im\left[S_{xy}(\omega)\right] \quad [\text{By } (1)]$$

$\therefore Im\left[S_{xy}(\omega)\right]$ is an odd function of ω.

J. The cross power spectrum of a real random process $X(t)$ and $Y(t)$ is given by,

$$S_{xy}(\omega) = \begin{cases} a + \frac{ib\omega}{\alpha}, & -\alpha < \omega < \alpha,\ \alpha > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the cross correlation function.

Soln.

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\left(a + \frac{ib\omega}{\alpha}\right) e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} a e^{i\omega\tau}\, d\omega + \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{ib\omega}{\alpha} e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\alpha}^{\alpha} + \frac{ib}{2\pi\alpha}\left[\omega\left(\frac{e^{i\omega\tau}}{i\tau}\right) - \left(\frac{e^{i\omega\tau}}{(i\tau)^2}\right)\right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{2\pi i\tau}\left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right] + \frac{ib}{2\pi\alpha}\left[\left(\frac{\alpha}{i\tau}e^{i\alpha\tau} + \frac{e^{i\alpha\tau}}{\tau^2}\right) - \left(-\frac{\alpha e^{-i\alpha\tau}}{i\tau} + \frac{e^{-i\alpha\tau}}{\tau^2}\right)\right]$$

$$= \frac{a}{2\pi i\tau}\left[2i \sin\alpha\tau\right] + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}\left(e^{i\alpha\tau} - e^{-i\alpha\tau}\right) + \frac{1}{\tau^2}\left(e^{i\alpha\tau} - e^{-i\alpha\tau}\right)\right]$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}\, 2\cos\alpha\tau + \frac{1}{\tau^2}\, 2i \sin\alpha\right.$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{b}{\pi\tau}\cos\alpha\tau - \frac{b}{\pi\alpha\tau^2}\sin\alpha\tau$$

2). If the cross correlation of two processes $\{X(t)\}$ and $\{Y(t)\}$ is $R_{XY}(t, t+\tau) =$

$$\frac{AB}{2}\left[\sin(\omega_0 \tau) + \cos(\omega_0(2t+\tau))\right]$$

where A, B are ω_0 and constants. Find the cross power spectrum.

Soln.

The time average is given by,

$$R_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{AB}{2}\left[\sin \omega_0 \tau + \cos \omega_0(2t+\tau)\right] d\tau$$

$$= \frac{AB}{2}\left[\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin(\omega_0 \tau)\, d\tau + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos\omega_0(2t+\tau)\, d\tau\right]$$

$$= \frac{AB}{2}\left[\sin \omega_0 \tau + \lim_{T\to\infty} \frac{1}{4T} \sin(\omega_0(2t+\tau))\right]_{-T}^{T}$$

$$= \frac{AB}{2} \sin \omega_0 \tau + 0$$

The cross spectrum is

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{AB}{2} \sin \omega_0 \tau\, e^{-i\omega\tau}\, d\tau$$

$$= \frac{AB}{2}\int_{-\infty}^{\infty} \sin \omega_0 \tau\,(\cos \omega\tau - i \sin \omega\tau)\, d\tau$$

$$= \frac{AB}{2}\left[\int_{-\infty}^{\infty} \sin \omega_0 \tau \cos \omega \tau \, d\tau - i \int_{-\infty}^{\infty} \sin \omega_0 \tau \sin \omega \tau \, d\tau\right]$$

$$= \frac{AB}{2}\left[\frac{1}{2}\int_{-\infty}^{\infty}\left[\sin(\omega_0+\omega)\tau + \sin(\omega_0-\omega)\tau\right] d\tau - \frac{i}{2}\int_{-\infty}^{\infty}\left[\cos(\omega-\omega_0)\tau - \cos(\omega+\omega_0)\tau\right] d\tau\right]$$

$$= \frac{AB}{4}\left[\int_{-\infty}^{\infty} \sin(\omega_0+\omega)\tau + \sin(\omega_0-\omega)\tau - i\cos(\omega-\omega_0)\tau + i\cos(\omega+\omega_0)\tau\right] d\tau\Big]$$

$$= \frac{AB}{4}\int_{-\infty}^{\infty}\Big\{ i\left[\cos(\omega-\omega_0)\tau - i\sin(\omega-\omega_0)\tau\right] - i\left[\cos(\omega-\omega_0)\tau - i\sin(\omega-\omega_0)\tau\right]\Big\} d\tau$$

$$= -i\frac{AB}{4}\int_{-\infty}^{\infty}\left[e^{-i(\omega-\omega_0)\tau} - e^{-i(\omega+\omega_0)\tau}\right] d\tau$$

$$= -i\frac{AB}{4}\left[\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)\tau} d\tau - \int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)\tau} d\tau\right]$$

$$= -\frac{iAB}{4}\left[2\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0)\right]$$

$$= -\frac{i\pi AB}{2}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

3. Given the Power Spectral density $S_{XX}(\omega) = \frac{\omega^2+9}{\omega^4+5\omega^2+4}$. find the mean square value of the process.

Soln.

Given $S_{XX}(\omega) = \frac{\omega^2+9}{\omega^4+5\omega^2+4}$

put $u = \omega^2$

$$S_{XX}(\omega) = \frac{u+9}{u^2+5u+4} = \frac{u+9}{(u+1)(u+4)} = \frac{A}{u+1} + \frac{B}{u+4}$$

$$u+9 = A(u+4) + B(u+1)$$

put $u=-1$, $-1+9 = A(-1+4)+0 \Rightarrow A = 8/3$

$u=-4$, $-4+9 = 0 + B(-4+1) \Rightarrow B = -5/3$

$$\therefore S_{XX}(\omega) = \frac{8}{3}\frac{1}{u+1} - \frac{5}{3}\frac{1}{u+4} = \frac{8}{3}\frac{1}{\omega^2+1} - \frac{5}{3}\frac{1}{\omega^2+4}$$

To find :

Mean square value of the process i.e., $R_{XX}(0)$

WKT

$$R_{XX}(\tau) = F^{-1}(S_{XX}(\omega))$$

$$= F^{-1}\left[\frac{8}{3}\frac{1}{\omega^2+1} - \frac{5}{3}\frac{1}{\omega^2+4}\right]$$

$$= \frac{8}{3}\cdot\frac{1}{2}F^{-1}\left(\frac{2}{\omega^2+1^2}\right) - \frac{5}{3}\cdot\frac{1}{4}F^{-1}\left(\frac{2\cdot 2}{\omega^2+2^2}\right)$$

$$= \frac{8}{6}e^{-|\tau|} - \frac{5}{12}e^{-2|\tau|}$$

Now, $R_{XX}(0) = \frac{8}{6} - \frac{5}{12}$

$$= \frac{11}{12}$$