Bilinear Transformation:

The transformation $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$

where a, b, c, d are complex numbers is called a bilinear transformation.

It is also called as Mobius transformation (or) linear fractional transformation.

Fixed Points (or) Invariant Points:

The fixed points of the transformation $w = \frac{za+b}{cz+d}$

is obtained from $z = \frac{az+b}{cz+d}$ (or) $cz^2+(d-a)z-b=0$.

Note:

1. The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

2. The cross ratio of four points

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Problems:

1) Find the fixed points of $w = \frac{2zi+5}{z-4i}$.

Sol: The fixed points are $z = \frac{2zi+5}{z-4i}$

$$z^2-4zi = 2zi+5$$

$z^2 - 6zi - 5 = 0$

$z = \frac{6i \pm \sqrt{-36+20}}{2}$

$z = 5i, i$

2) Find the invariant points of the bilinear transformation $\frac{z-1}{z+1}$.

Sol:. The invariant points are given by $z = \frac{z-1}{z+1}$

$z^2 + z = z - 1$

$z^2 = -1$

$z = \pm\sqrt{-1}$

$= i, -i.$

3) Obtain the invariant points of the transformation $w = 2 - \frac{2}{z}$

Sol:. The invariant points are given by

$z = 2 - \frac{2}{z}$, $z = \frac{2z-2}{z}$

$z^2 = 2z - 2$, $z^2 - 2z + 2 = 0$

$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$

4) Find the fixed points :.

i) $w = \frac{z-1-i}{z+2}$ Ans:. $z = \frac{-1 \pm i\sqrt{3+4i}}{2}$

ii) $w = \frac{2z-5}{z+4}$ Ans: $-1 \pm 2i$

iii) $w = \frac{z-2}{z+3}$ Ans:. $z = -1 \pm i$

iv) $w = \frac{1}{z-2i}$ Ans: $z = i$

v) $w = \frac{5z+4}{z+5}$ Ans: $z = \pm 2$.

Problems:

1) Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively.

Sol: Given: $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ and $w_1 = 0$, $w_2 = i$, $w_3 = \infty$.

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\,z_1\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)} = \frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{w}{i} = \frac{i}{z}$$

$$wz = i^2$$

$$wz = -1$$

$$\boxed{w = \frac{-1}{z}}$$

2) Determine the bilinear transformation that maps the points $-1, 0, 1$ in the z-plane onto the points $0, i, 3i$ in a w-plane.

Sol: Given $z_1 = -1,\ z_2 = 0,\ z_3 = 1$

$w_1 = 0,\ w_2 = i,\ w_3 = 3i$

Let the transformation be $\dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$-\frac{2w}{w-3i} = -\frac{z+1}{z-1}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2wz - 2w = wz + w - 3iz - 3i$$

$$2wz - 2w - wz - w = -3i(z+1)$$

$$w(2z-2-z-1) = -3i(z+1)$$

$$w(z-3) = -3i(z+1)$$

$$\boxed{w = -3i\frac{(z+1)}{(z-3)}}$$

3) Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, i, -i$ respectively.

Sol: Given: $z_1 = -2,\ z_2 = 0,\ z_3 = 2$

$w_1 = 0,\ w_2 = i,\ w_3 = -i$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-(-i))}{(w-(-i))(i-0)} = \frac{(z-(-2))(0-2)}{(z-2)(0-(-2))}$$

$$\frac{w(2i)}{(w+i)(i)} = \frac{(z+2)(-2)}{(z-2)(2)}$$

$$\frac{2wi}{(w+i)(i)} = \frac{-2z-4}{2z-4}$$

$$\frac{2w}{w+i} = -\frac{(z+2)}{(z-2)}$$

$$2wz-4w = -zw-zi-2w-2i$$

$$2wz-4w+zw+2w = -zi-2i$$

$$w(2z-4+z+2) = -i(z+2)$$

$$w(3z-2) = -i(z+2)$$

$$\boxed{w = \frac{-i(z+2)}{3z-2}}$$

4) Find the bilinear transformation which maps the Points $1, i, -1$ onto the Points $0, 1, \infty$. show that the transformation maps the interior of the unit circle of the z-plane onto the upper half of the w plane.

Sol: Given $z_1 = 1,\ z_2 = i,\ z_3 = -1$

$w_1 = 0,\ w_2 = 1,\ w_3 = \infty$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)w_3}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w = i\,\frac{(z-1)}{(z+1)}.$$

Thus the region $|z|<1$ gives onto the region $\left|\frac{w-i}{w+i}\right|<1$.

Let $w=u+iv$ we get $\left|\frac{u+iv-i}{u+iv+i}\right|<1$

$$|u+iv-i|<|u+iv+i|$$

$$|u+i(v-1)|<|u+i(v+1)|$$

$$u^2+(v-1)^2<u^2+(v+1)^2$$

$$(v-1)^2<(v+1)^2$$

$v^2 - 2v + 1 < v^2 + 2v + 1$

$v^2 - 2v + 1 - v^2 - 2v - 1 < 0$

$-4v < 0$

i.e., $v > 0$

The interior of unit circle is mapped onto the upper half of the w plane.

5) $0, i, \infty$ onto $\infty, i, 0$

6) $z_1 = 2, z_2 = i, z_3 = -2$, into $w_1 = 1, w_2 = i, w_3 = -1$.