Cross correlation :

A cross correlation of a two Random Processes $X(t)$ & $Y(t)$ is defined by,

$$R_{XY}(\tau) = E[X(t)\ Y(t+\tau)]$$

$$R_{YX}(\tau) = E[Y(t) \cdot X(t+\tau)]$$

1. If $X(t)$ and $Y(t)$ are WSS process, then $R_{XY}(-\tau) = R_{YX}(\tau)$

Proof :

WKT $R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$

Now, $R_{XY}(-\tau) = E[X(t)\ Y(t-\tau)]$

Take $t-\tau = t_1 \Rightarrow t = t_1+\tau$

$$R_{XY}(-\tau) = E[X(t_1+\tau)\ Y(t_1)]$$
$$= E[Y(t_1)\ X(t_1+\tau)]$$
$$R_{XY}(-\tau) = R_{YX}(\tau)$$

2. If $X(t)$ and $Y(t)$ are WSS process, then

$$R_{XY}(\tau) \le \sqrt{R_{XX}(0)\, R_{YY}(0)}$$

Proof :

By cross correlation,

$$R_{XY}(\tau) = E[X(t)\ Y(t+\tau)]$$

By Schwartz inequality property,

$$[E(XY)]^2 \le E(X^2) \cdot E(Y^2)$$

Let $X = X(t)$, $Y = Y(t+\tau)$

$$E[X(t)\ Y(t+\tau)]^2 \le E[X^2(t)] \cdot E[Y^2(t+\tau)]$$

$$[R_{XY}(\tau)]^2 \le E[X^2(t)]\ E[Y^2(t+\tau)]$$

$$\because E[Y^2(t+\tau)] = E[Y^2(t)]$$

$$[R_{xy}(\tau)]^2 \le E[x^2(t)] \cdot E[y^2(t)]$$

By Prop. (1), $\Rightarrow$ $E[x^2(t)] = R_{xx}(0)$

$E[y^2(t)] = R_{yy}(0)$

$$\therefore [R_{xy}(\tau)]^2 \le R_{xx}(0) \cdot R_{yy}(0)$$

$$R_{xy}(\tau) \le \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$$

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3. Consider two R.P. $x(t) = 3\cos(\omega t + \theta)$ and $y(t) = 2\cos(\omega t + \phi)$, where $\phi = \theta - \frac{\pi}{2}$ and θ is uniformly distributed random variable over $(0, 2\pi)$. verify that $|R_{xy}(\tau)| \le \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$

Soln.

PDF

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Then $R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$

$$= E[9\cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)]$$

$$= \frac{9}{2} E[2\cos(\omega t + \omega\tau + \theta) \cdot \cos(\omega t + \theta)]$$

$$= \frac{9}{2} E[\cos \omega\tau] + \frac{9}{2} E[\cos(2\omega t + \omega\tau + 2\theta)]$$

$$= \frac{9}{2}\cos\omega\tau + \frac{9}{4\pi}\int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta)\, d\theta$$

$$= \frac{9}{2}\cos\omega\tau + \frac{9}{8\pi}\left[\sin(2\omega t + \omega\tau + 2\theta)\right]_0^{2\pi}$$

$$R_{xx}(\tau) = \frac{9}{2}\cos\omega\tau.$$

$$Y(t) = 2\cos(\omega t + \phi)$$
$$= 2\cos(\omega t + \theta - \pi/2)$$
$$= 2\cos[\pi/2 - (\omega t + \theta)]$$
$$Y(t) = 2\sin(\omega t + \theta)$$

$$\therefore R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)]$$
$$= 2E[2\sin(\omega t + \theta)\sin(\omega t + \omega\tau + \theta)]$$
$$= 2\left[E[\cos\omega\tau] - E[\cos(2\omega t + \omega\tau + 2\theta)]\right]$$
$$= 2\cos\omega\tau - 0 \qquad \because E[\cos(2\omega t + \omega\tau + 2\theta)] = 0$$
$$R_{YY}(\tau) = 2\cos\omega\tau$$

$$R_{XY}(t, t+\tau) = E[X(t)\cdot Y(t+\tau)]$$
$$= E[6\cos(\omega t + \theta)\sin(\omega t + \omega\tau + \theta)]$$
$$= 3E[2\cos(\omega t + \theta)\sin(\omega t + \omega\tau + \theta)]$$
$$= 3E[\sin(2\omega t + \omega\tau + 2\theta) + \sin\omega\tau]$$
$$= 3\{E[\sin(2\omega t + \omega\tau + 2\theta)] + E[\sin(\omega\tau)]\}$$
$$R_{XY}(\tau) = 3\sin\omega\tau$$

$$|R_{XY}(\tau)| = |3\sin\omega\tau| \le 3 \qquad \because |\sin\omega t| \le 1$$

and $R_{XX}(0)\cdot R_{YY}(0) = \frac{9}{2}\cdot 2 = 9$

$$\sqrt{R_{XX}(0)\cdot R_{YY}(0)} = 3$$

$$\therefore |R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\cdot R_{YY}(0)}$$

Let $X(t)$ & $Y(t)$ is defined by $X(t) = A\cos\omega t + B\sin\omega t$ and $Y(t) = B\cos\omega t - A\sin\omega t$ where ω is a constant and A, B are independent r.vs. both having zero mean & variance σ^2. Find the cross correlation of $X(t)$ & $Y(t)$. Are $X(t)$ & $Y(t)$ jointly WSS process?