Wiener - Khintchine Theorem:

If $X_T(\omega)$ is the FT of the truncated random process is defined as,

$$X_T(t) = \begin{cases} X(t) & \text{for } |t| \le T \\ 0 & \text{for } |t| > T \end{cases}$$

where $[X(t)]$ is a real WSS process with power spectral density function $S_{XX}(\omega)$, then

$$S_{XX}(\omega) = \lim_{T \to \infty} \left[\frac{1}{2T} E\{ |X_T(\omega)|^2 \} \right]$$

Proof:

Given $X_T(\omega)$ is the Fourier transform of $X_T(t)$

$$\Rightarrow \quad X_T(\omega) = \int_{-\infty}^{\infty} X_T(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-T}^{T} X(t)\, e^{-i\omega t}\, dt$$

Now, $\quad |X_T(\omega)|^2 = [X_T(\omega)]\overline{[X_T(\omega)]} \qquad \because |z|^2 = z\bar{z}$

$$= X_T(\omega)\, X_T(-\omega)$$

$$= \int_{-T}^{T} X(t_1)\, e^{-i\omega t_1}\, dt_1 \cdot \int_{-T}^{T} X(t_2)\, e^{i\omega t_2}\, dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} X(t_1)\, X(t_2)\, e^{-i\omega(t_2 - t_1)}\, dt_2\, dt_1$$

$$[|X_T(\omega)|^2] = \int_{-T}^{T}\int_{-T}^{T} E[X(t_1)\, X(t_2)]\, e^{-i\omega(t_2 - t_1)}\, dt_2\, dt_1$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1, t_2)\, e^{-i\omega(t_2 - t_1)}\, dt_2\, dt_1 \rightarrow (1)$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)\, e^{-i\omega(t_1 - t_2)}\, dt_1\, dt_2$$

Let $t = t_2 \Rightarrow \tau = t_1 - t \Rightarrow \tau + t = t_1$

$dt = dt_2$ $\qquad d\tau = dt_1$

$$E[|X_T(\omega)|^2] = \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)\, e^{-i\omega(t_1 - t_2)}\, dt_1\, dt_2$$

t varies from $-T$ to T &

τ varies from $-T$ to T

$\tau = -T \Rightarrow t_1 - t = -T \Rightarrow t_1 = -T + t$

$\tau = T \Rightarrow t_1 - t = T \Rightarrow t_1 = T + t$

$$E[|X_T(\omega)|^2] = \int_{-T}^{T}\int_{-T+t}^{T+t} R_{XX}(t_1 - t_2)\, e^{-i\omega(t_1 - t_2)}\, dt\, d\tau$$

$$= \int_{-T}^{T} dt \int_{-T+t}^{T+t} R_{XX}(t_1 - t_2)\, e^{-i\omega(t_1 - t_2)}\, d\tau$$

$$= (t)_{-T}^{T} \int_{-T+t}^{T+t} R_{XX}(t_1 - t_2)\, e^{-i\omega(t_1 - t_2)}\, d\tau$$

$$= 2T \int_{-T+t}^{T+t} R_{XX}(t_1 - t_2)\, e^{-i\omega(t_1 - t_2)}\, d\tau$$

$$\lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(\omega)|^2\right] = \lim_{T\to\infty} \frac{1}{2T}(2T)\int_{-T+t}^{T+t} R_{XX}(t_1-t_2)\, e^{-i\omega(t_1-t_2)}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(t_1-t_2)\, e^{-i\omega(t_1-t_2)}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= S_{XX}(\omega)$$

Hence proved