SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore – 641 035

DEPARTMENT OF MATHEMATICS

23MAT203-PROBABILITY AND RANDOM PROCESSES

UNIT – 1
PROBABILITY AND RANDOM VARIABLES
Two marks

1. Define i) Discrete random variable

   ii) Continuous random variable

   i) Let X be a random variable, if the number of possible values of X is finite or count ably finite, then X is called a discrete random variable.

   ii) A random variable X is called the continuous random variable, if x takes all its possible values in an interval.

2. Define probability mass function (PMF):

   Let X be the discrete random variable taking the values ,X_1 , X_2 ……… Then the number P (X_i) = P(X =X_i) is called the probability mass function of X and it satisfies the following conditions.

   i) P (X_i) $\geq$ 0 for all;

   ii) $\sum_{i=1}^{\infty}$ P (X_i) = 1

3. Define probability Density function (PDF):

   Let x be a continuous random variable. The Function f(x) is called the probability density function (PDF) of the random variable x if it satisfies.

   i) f(x) $\geq$ 0

   ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

4. Define cumulative distribution function (CDF):

   Let x be a random variable. The cumulative distribution function, denoted by F(X)and is given by F(X)=P(X≤x)

5. If x is a discrete R.V having the p.m.f

X:	-1	0	1
P(X):	K	2k	3k

Find P(x≥ 0)

Answer: $6k = 1 \Rightarrow k = \frac{1}{6}$

$P[x \geq 0] = 2k + 3k \Rightarrow P[x \geq 0] = \frac{1}{6}$

6. The random variable x has the p.m.f. P (x)=$\frac{x}{15}$, x=1,2,3,4,5 and = 0 else where.

Find P $[\frac{1}{2} < x < \frac{5}{2} / x > 1]$.**Answer:**

$$P\left[\frac{1}{2} < x < \frac{5}{2} / x > 1\right] = \frac{P[x=2]}{P(x>1)} = \frac{P[x=2]}{1-P(x \leq 1)} = \frac{2/15}{1-1/15} = \frac{1}{7}$$

7. If the probability distribution of X is given as :

X	1	2	3	4
P(X)	0.4	0.3	0.2	0.1

Find P $[\frac{1}{2} < x < \frac{7}{2} / x > 1]$.

Answer :

$$P\left[\frac{1}{2} < x < \frac{7}{2} / x > 1\right] = \frac{P[1<x<7/2]}{P(x>1)} = \frac{P(x=2)+P(x=3)}{1-P(x=1)} = \frac{0.5}{0.6} = \frac{5}{6}$$

8. A.R.V. X has the probability function

X	-2	-1	0	1
P(X)	0.4	k	0.2	0.3

Find k and the mean value of X

Answer:

k=0.1 Mean $=\sum xP(x)$ = $\frac{1}{10}$ [-8-1+0+3] = -0.6

9. If the p.d.fof a R.V. X is f(x) = $\frac{x}{2}$ in $0 \le x \le 2$, find

P$[x > 1.5/x > 1]$.

Answer :

P$[x > 1.5/x > 1] = \frac{P[x>1.5]}{P(x>1)} = \frac{\int_{1.5}^{2} \frac{x}{2} dx}{\int_{1}^{2} \frac{x}{2} dx} = \frac{4-2.25}{4-1} = 0.5833$

10.If the p.d.f of a R.V.X is given by f(x) = {1/4,-2<x<2.0, else where. Find P[|X|>1].

Answer:

P[|X|>1] =1- P[|X|<1] = 1-$\int_{-1}^{1} \frac{1}{4} dx$ = $\frac{1}{2}$

11. If f(x) = kx^2, 0<x<3 is to be density function, Find the value of k.

Answer:

$\int_0^3 kx^2 dx$= 1 ⇒ 9k = 1 ∴ k = $\frac{1}{9}$

12. If the c.d.f. of a R.V X is given by F(x) = 0 for x<0;= $\frac{x^2}{16}$for $0 \le x < 4 \quad and =$ $1 \; for \; x \; \ge 4, find \;\; P(X > 1/X < 3)$.

Answer:

$$P(X > 1/X < 3) = \frac{P[1<X<3]}{P[0<X<3]} = \frac{F(3)-F(1)}{F(3)-F(0)} = \frac{8/16}{9/16} = \frac{8}{9}.$$

13.The cumulative distribution of X is F(x) $= \frac{x^3+1}{9}$ $, -1, < X < 2$ $and \quad =$ 0, otherwise. Find P[0 <X<1].

Answer:

P[0<X<1] = F(1) – F(0) = $\frac{2}{9} - \frac{1}{9} = \frac{1}{9}$

14. A Continuous R.V X that can assume any value between x=2 and x=5 had the p.d.f f(x) = k(1+x). Find P(x<4).

Answer:

$$\int_2^3 k(1+x)dx = 1 \Rightarrow \frac{27k}{2} = 1 \therefore k = \frac{2}{27}$$

P[X<4] $=\int_2^4 \frac{2}{27}(1+x)dx = \frac{16}{27}$

15. The c.d.f of X is given by F (x) = $\begin{cases} 0, x > 0 \\ x^2, \ 0 \le x \le 1 \\ 1, x > 1 \end{cases}$ Find the p.d.f of x, and obtain P(X>0.75).

Answer:

F(x) = $\frac{d}{dx}$F(x) = $\begin{cases} 2x, 0 \le x \le 1 \\ 0, otherwise \end{cases}$

P[x<0.75] = 1- P[X≤ $0.75] = 1 - F(0.75) = 1 - (0.75)^2 = 0.4375$

16. Check whether f(x)=$\frac{1}{4}xe^{-x/2}$for 0<x<∞ can be the p.d.f of X.

Answer:

$= \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{x}{4} e^{-x/2}\, dx = \int_{0}^{\infty} te^{-1}\, dt$ where t = $\frac{x}{2}$

=$(-te^{-1} - e^{-1})_0^{\infty}$ = - [0-1]=1

$$\therefore f(x) is\ the\ p.d.f\ of\ X.$$

17. A continuous R.V X has a p.d.f f(x) = $3x^2, 0 \le x \le 1$. Find b such that P(X>b)= 0.05.

Answer:

$$3\int_{b}^{1} x^2 dx = 0.05 \Rightarrow 1 - b^3 = 0.05 \Rightarrow b^3 = 0.95 \ \therefore b = (0.95)^{\frac{1}{3}}$$

18. Let X be a random variable taking values -1, 0 and 1 such that P(X=-1) = 2P(X=0) = P(X=1). Find the mean of 2X-5.

Answer:

$$\sum P(X = x) = 1 \Rightarrow 5P(X = 0) = 1 \ \therefore P(X = 0) = \frac{1}{5}$$

Probability distribution of X:

X	-1	0	1
P(X)	2/5	1/5	2/5

$$Mean = E(x) = \sum xp(x) = -1\left(\frac{2}{5}\right) + 0\left(\frac{1}{5}\right) + 1\left(\frac{2}{5}\right) = 0$$

E[2X-5] = 2E(X) – 5 = 2[0] – 5 = -5.

19. Find the cumulative distribution function F(x) corresponding to the p.d.f.

$F(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$.

Answer

$F(x) = \int_{-\infty}^{x} f(x)dx = \frac{1}{\pi}\int_{-\infty}^{x} \frac{dx}{1+x^2} = \frac{1}{\pi}[tan^{-1}x]$

$= \frac{1}{\pi}[\frac{\pi}{2} + tan^{-1} x]$

20. The diameter of an electric cable, say X is assumed to a continues R.V with p.d.f of given by f(x) = kx(1-x), $0 \le x \le 1$. Determine k and $P\left(x \le \frac{1}{3}\right)$

Answer:

$$\int_0^1 kx(1-x)dx = 1 \Rightarrow k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \quad \therefore k = 6$$

$$P\left[X \le \frac{1}{3}\right] = 6\int_0^{1/3}(x - x^2)dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1/3} = [(3x^2 - 2x^3)]_0^{1/3} = \frac{1}{3} - \frac{2}{27} = \frac{7}{27}$$

21. A random variable X has the p.d.f f(x) given by $f(x) = \begin{cases} Cxe^{-x}, if\ x > 0 \\ 0, if\ x \le 0 \end{cases}$. Find the value of C and C.D.F of X.

Answer:

$$C\int_0^\infty xe^{-x}dx = 1 \Rightarrow C[x(-e^{-x}]_0^\infty = 1$$

$$\therefore C[-0 + 1] = 1 \Rightarrow C = 1$$

$C.D.F : F(x) = \int_0^x f(x)dx = 1 - (1 + x)e^{-x}$ for $x \ge 0$.

22. State the properties of cumulative distribution function.

Answer:

i) F(-∞)=0 and F(∞) $= 1$.

ii) F(∞) is non – decreasing function of X.

iii) If F(∞)$is\ the\ p.d.f$ of X, then f(x)=$F'(x)$

iv) P[a≤ $X \leq b$] = F(b) – F(a)

23. Define the raw and central moments of R.Vand state the relation between them.

Answer:

Raw moment μ'_r = E[X^r]

Central momentμ_r= E[$\{X - E(X)\}^r$].

$\mu_r = \mu'_r$- r$C_1\mu'_{r-1}\,\mu'_r$+ r$C_2\mu'_{r-2}(\mu'_r)^2$ - …….+$(-1)^r(\mu'_1)^r$

24.The first three moments of a R.V.X about 2 are 1, 16, -40. Find the mean, variance of X. Hence find μ_3.

Answer:

E(X) = $\mu'_1 + A \Rightarrow Mean = 1 + 2 = 3$

Variance = E(X^2)–[E((X)]$^2 = 16 - 1 = 15$

$\mu_3 = \mu'_3$- $3\mu'_2\mu'_1 + 2(\mu'_1)^3 = -86$

25. Find the r-th moment about origin of the R.V X with p.d.f$f(x) =$

$\begin{cases} Ce^{-ax}, x \geq 0 \\ 0, \ else\ where \end{cases}$

Answer:

$$\int_0^\infty C\, e^{-ax} dx = 1 \Rightarrow C = a$$

$$\mu'_r = \int_0^\infty x^r f(x) dx = a \int_0^\infty x^{(r+1)-1} e^{-ax} dx = \frac{\sqrt{(r+1)}}{a^r} = \frac{r!}{a^r}$$

26. A C.R.V X has the p.d.f f(x)=$kx^2e^{-x}, x > 0$. Find the r-th moment about the origin.

Answer:

$$\int_0^\infty kx^2 e^{-x} dx = 1 \Rightarrow k = \frac{1}{2}$$

$$\mu'_1 = E[X^r] = \frac{1}{2}\int_0^\infty x^{r+2} e^{-x}\, dx = \frac{1}{2}\sqrt{(r+3)} = \frac{(r+2)!}{2}$$

27.If X and Y are independent R,V's and Z = X+Y, prove that $M_x(t)M_y(t)$.

Answer:

$$M_z(t) = E[e^{tz}] = E\left[e^{t(X+Y)}\right] = E[e^{tx}]E[e^{ty}]$$

$$= M_x(t)M_y(t).$$

28.If the MGF of X is $M_x(t)$and if Y=aX+b show that $M_y(t) = e^{bt}M_x(at)$.

Answer:

$$M_y(t) = E[e^{ty}] = E[e^{bt}e^{axt}] = e^{bt}E[e^{(at)X}] = e^{bt}M_x(at).$$

29.If a R.V X has the MGF M(t)=$\frac{3}{3-t}$, obtain the mean and variance of X.

Answer:

M(t) = $\frac{3}{3[1-\frac{t}{3}]}$ =1+$\frac{t}{3} + \frac{t^2}{9} + \dots$..

E(x) = Co-efficient of $\frac{t}{1!}$in (1) = $\frac{1}{3}$

E(X^2) = co-efficient of $\frac{t^2}{2!}$in (1) = $\frac{1}{9}$

∴ Mean = $\frac{1}{3}$ and V(X) = E(X^2) – [E(X)]^2 = $\frac{1}{9}$

30.If the r-th moment of a C.R.V X about the origin is r!, find the M.G. F of X.

Answer:

$$M_x(t) = \sum_{r=0}^{\infty} E[X^r]\ .\frac{t^r}{r!} = \sum_{r=0}^{\infty} t^r$$

$$= 1 + t + \ t^2 + \cdots\ \ = (1-t)^{-1} = \frac{1}{1-t}$$

31.If the MGF of a R.V. X is $\frac{2}{2-t}$, Find the standard deviation of x.

Answer:

$M_x(t)$ = $\frac{2}{2-t}$ = $(1-\frac{t}{2})^{-1} = 1 + \frac{t}{2} + \frac{t^2}{4} + \cdots$

$E(X)$ = $\frac{1}{2}$; E(x^2)= $\frac{1}{2}$; V(X) = $\frac{1}{4}$ ⇒ $S.D\ of\ X = \frac{1}{2}$

32.Find the M.G.F of the R.V X having p.d.f$f(x) = \begin{cases} \frac{1}{3}, -1 < x < 2 \\ 0,\ \ else\ where \end{cases}$

Answer:

$M_x(t)$ = $\int_{-1}^{2} \frac{1}{3} e^{tx} dx$ = $\frac{1}{3t}[e^{2t} - e^{-t}]\ for\ t \neq 0$

When t=0 , $M_x(t) = \int_{-1}^{2} \frac{1}{3}\ dx = 1$

$$\therefore M_x(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t}, t \neq 0 \\ 1, t = 0 \end{cases}$$

33.Find the MGF of a R.V X whose moments are given by $\mu'_r = (r = 1)!$

Answer:

$$M_x(t) = \sum_{r=0}^{\infty} E[X^r] \cdot \frac{t^r}{r!} = \sum_{r=0}^{\infty} (r+1)t^r$$

$$= 1 + 2t + 3t^2 + \cdots = (1-t)^{-2}$$

$$\therefore M_x(t) = \frac{1}{(1-t)^2}$$

34.Give an example to show that if p.d.f exists but M.G.F. does not exist.

Answer:

P(x)= $\begin{bmatrix} \frac{6}{\pi^2 x^2}, x = 1,2,\ldots \\ 0, otherwise \end{bmatrix}$

$$\sum P(x) = \frac{6}{\pi^2} \Rightarrow \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{6}{\pi^2}\left[\frac{\pi^2}{6}\right] = 1$$

∴ P(x) is a p.d.f.

But $M_x(t) = \frac{6}{\pi^2}\sum \frac{e^{tx}}{x^2}$, which is a divergent series

$$\therefore M_x(t)\ doesnt\ exist.$$

35.The moment generating function of a random variable X is given by $M_x(t) = \frac{1}{3}e^t + \frac{4}{15}e^{3t} + \frac{2}{15}e^{4t} + \frac{4}{15}e^{5t}$.Find the probability density function of X.

Answer:

X	1	2	3	4
P(X)	1/3	4/15	2/15	4/15

36.Let $M_x(t)\frac{1}{(1-t)}, t < 1$ be the M.G.F of a R.V X. Find the MGF of the RV Y=2X+1.

Answer:

If Y =aX+b, $M_y(t) = e^{bt}M_x(at)\ \therefore M_y(t) = \frac{e^t}{1-2t}$.

37.Suppose the MGF of a RV X is of the form $M_x(t) = (0.4e^t + 0.6)^8$.What is the MGF of the random variable Y=3X+2.

Answer:

$$M_y(t) = e^{2t}M_x(3t) = e^{2t}[(0.4)e^{3t} = 0.6)]^8$$

38.The moment generating function of a RV X is $\left[\frac{1}{5} + \frac{4e^t}{5}\right]^{15}$. Find the MGF of Y=2X + 3.

Answer:

If Y = 2X + 3, then $M_y(t) = e^{3t}M_x(2t)$.

$$\therefore M_y(t) = e^{3t}\left[\frac{1}{5} + \frac{4e^t}{5}\right]^{15}$$

39. If a random variable takes the values -1, 0 and 1 with equal probabilities, find the MGF of X.

Answer:

$M_x(t) = \sum \ e^{tx}\, P(x) = \frac{1}{3}e^{-1} + \frac{1}{3} + \frac{1}{3}e^{1} = \frac{1}{3}[1 + e^1 + e^{-1}]$