SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore – 641 035

DEPARTMENT OF MATHEMATICS

23MAT203-PROBABILITY AND RANDOM PROCESSES

UNIT 2

TWO DIMENSIONAL RANDOM VARIABLES

Two marks

1. The bivariate random variable X and Y has the pdf $f(x,y)=\begin{cases} kx^2(8-y), x<y<2x \\ \qquad 0\le x<2 \end{cases}$

find k.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dydx=1$$

$$\int_{0}^{2}\int_{x}^{2x} kx^2(8-y)\,dydx=1 \qquad k\int_{0}^{2} x^2\left(8y-\frac{y^2}{2}\right)_{x}^{2x}dx=1$$

$$k\int_{0}^{2} x^2\left(16x-\frac{4x^2}{2}-8x+\frac{x^2}{2}\right)dx=1 \quad k\int_{0}^{2}\left(16x^3-2x^4-8x^3+\frac{x^4}{2}\right)dx=1$$

$$k\int_{0}^{2}\left(8x^3-\frac{3x^4}{2}\right)dx=1 \quad k\left[\frac{8x^4}{4}-\frac{3x^5}{10}\right]_0^2=1$$

$$k\left[32-\frac{48}{5}\right]=1 \qquad k\left[\frac{112}{5}\right]=1$$

$$k\left[\frac{112}{5}\right]=1$$

$$k=\frac{5}{112}$$

2. The joint pdf of random variable x and y is given by $f(x,y)=kxye^{-(x^2+y^2)}, x>0, y>0$ find the value of k.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy=1$$

$$\int_0^\infty\int_0^\infty kxye^{-(x^2+y^2)}dydx=1$$

$$k\int_0^\infty ye^{-y^2}dy\int_0^\infty xe^{-x^2}dx=1 \qquad \left[\int_0^\infty xe^{-x^2}dx=\frac{1}{2}\right]$$

$$k\frac{1}{2}.\frac{1}{2}=1, \quad k=4$$

3. If X and Y have joint pdf $f(x,y)=\begin{cases} x+y, 0<x<1, 0<y<1 \\ 0, \quad \text{otherwise} \end{cases}$. check whether X and Y are independent.

   Ans:

   The marginal density of X is

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy \qquad f(x)=\int_0^1 (x+y)dy$$

$$f(x)=\left[xy+\frac{y^2}{2}\right]_0^1 \qquad f(x)=x+\frac{1}{2}$$

$$f(y)=\int_{-\infty}^{\infty} f(x,y)dy \qquad f(y)=\int_0^1 (x+y)dy$$

$$f(y)=\left[\frac{x^2}{2}+xy\right]_0^1 \qquad f(y)=\frac{1}{2}+y$$

$$f(x).f(y)=\left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \qquad f(x).f(y)\neq f(x,y)$$

4. Let X and Y have j.d.f f(x,y)=2, 0<x<y<1. Find m.d.f

   Ans:

   Marginal density of X is given by

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_{x}^{1} 2dy$$

$$=2[y]_x^1$$

$$=2(1-x), 0<x<1.$$

Marginal density function of Y is given by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)dy=\int_{0}^{y} 2dx=2[x]_0^y=2y, 0<y<1.$$

5. The j.d.f of the random variables X and Y is given by

$$f(x,y)=\begin{cases}8xy, 0<x<1, 0<y<x\\ 0, \quad \text{otherwise}\end{cases}. \text{find} f_x(x).$$

Ans:

$$f_x(x)=\int_{-\infty}^{\infty} f(x,y)dy$$

$$=\int_{0}^{x} 8xydy=8x\left(\frac{y^2}{2}\right)_0^x$$

$$=8x\left(\frac{x^2}{2}\right)$$

$$f_x(x)=4x^3, 0<x<1$$

6. Given $f(x,y)=\begin{cases}cx(x-y), 0<x<2, -x<y<x\\ 0, \quad \text{otherwise}\end{cases}$, find c.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx=1$$

$$\int_{0}^{2}\int_{-x}^{x} cx(x-y)dydx=1$$

$$c\int_{0}^{2}\left[x^2y-x.\frac{y^2}{2}\right]_{-x}^{x}dx=1$$

$$c\int_{0}^{2}\left[x^3-\frac{x^2}{2}+x^3+\frac{x^3}{2}\right]dx=1$$

$$c\int_{0}^{2}2x^3dx=1$$

$$2c\left[\frac{x^4}{4}\right]_0^2=1 \qquad 2c\left[\frac{16}{4}\right]=1 \qquad c=\frac{1}{8}$$

6. The joint p.d.f of a bivariate random variable (X,Y) is given by $f(x,y)=\begin{cases}kxy, 0<x<1, 0<y<1\\ 0, \quad otherwise\end{cases}$, find K.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy=1$$

$$\int_{0}^{1}\int_{0}^{1} kxydxdy=1 \qquad k\int_{0}^{1}\left[\frac{x^2}{2}y\right]_0^1dy=1$$

$$k\int_{0}^{1}\frac{y}{2}dy=1 \qquad k\left[\frac{y^2}{4}\right]_0^1=1$$

$$k=4$$

7. If the joint pdf of (x,y) is $\mathbf{f(x,y)=\frac{1}{4}, 0<x,y<1}$, find $\mathbf{p(x+y\le 1)}$.

Ans:

$$p(x+y\le 1)=p(x\le 1-y)$$

$$=\int_0^1\int_0^{1-y} f(x,y)\,dxdy$$

$$=\int_0^1\int_0^{1-y}\frac{1}{4}dxdy=\frac{1}{4}\int_0^1[x]_0^{1-y}dy$$

$$=\frac{1}{4}\int_0^1[(1-y)]dy \qquad =\frac{1}{4}\left[y-\frac{y^2}{2}\right]_0^1$$

$$=\frac{1}{4}\left[1-\frac{1}{2}\right] \qquad =\frac{1}{4}.\frac{1}{2} \quad =\frac{1}{8}$$

8. Two random variables X and Y have joint pdf $\mathbf{f(x,y)=\begin{cases}\frac{xy}{96}, 0<x<4, 1<y<5\\ 0, \quad otherwise\end{cases}}$, find E(x).

Ans:

$$E(x)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dxdy$$

$$=\int_1^5\int_0^4 x\left(\frac{xy}{96}\right)dxdy \qquad =\frac{1}{96}\int_1^5\left[y.\frac{x^3}{3}\right]_0^4 dy$$

$$=\frac{1}{96}\int_1^5\frac{64}{3}ydy \quad =\frac{64}{288}\left[\frac{y^2}{2}\right]_1^5$$

$$=\frac{2}{9}\left[\frac{25}{2}-\frac{1}{2}\right]=\frac{1}{9}(24)$$

$$=\frac{8}{3}.$$

9. Let X be a Random variable with pdf $f(x)=\frac{1}{2}, -1\leq x\leq 1$, and let $Y=X^2$, find E(Y).

Ans:

$Y=x^2$

$$E(Y)=E\left(x^2\right)$$

$$\int_{-\infty}^{\infty} x^2 f(x)dx=\int_{-\infty}^{\infty} x^2\left(\frac{1}{2}\right)dx=\frac{1}{2}\left(\frac{x^3}{3}\right)_{-1}^{1}=\frac{1}{6}(2)=\frac{1}{3}.$$

10. If the joint pdf of (x,y) is given by $f(x,y)=x+y, 0\leq x,y\leq 1$. Find $E(XY)$.

Ans:

$$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy$$

$$=\int_0^1\int_0^1 xy(x+y)dxdy$$

$$=\int_0^1\int_0^1\left(x^2y+xy^2\right)dxdy$$

$$=\int_0^1\left(\frac{x^3}{3}y+\frac{x^2}{2}y^2\right)_0^1 dy$$

$$=\int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)_0^1 dy$$

$$=\left(\frac{y^2}{6}+\frac{y^3}{6}\right)_0^1=\frac{2}{6}=\frac{1}{3}.$$

11. Find the acute angle between the two lines of regression

Ans:

The acute angle between the two lines of regression is $\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x{}^2+\sigma_y{}^2}\right)$.

12. State the equation of the two regression lines. What is the formula for correlation coefficient

Ans:

X on Y is $\left(\mathbf{x}-\bar{\mathbf{x}}\right)=\mathbf{b}_{\mathbf{xy}}\left(\mathbf{y}-\bar{\mathbf{y}}\right)$ and Y on X is $\left(\mathbf{y}-\bar{\mathbf{y}}\right)=\mathbf{b}_{\mathbf{yx}}\left(\mathbf{x}-\bar{\mathbf{x}}\right)$.

Correlation coefficient $\mathbf{r}=\sqrt{\mathbf{b}_{\mathbf{xy}}.\mathbf{b}_{\mathbf{yx}}}$.

13. If X and Y are independent random variables with variance 2 and 3. Find the variance of 3X+4Y.

Ans:

$$Var(x)=2,\ Var(y)=3$$

$$\begin{aligned}Var(3X+4Y) &= 3^2Var(X)+4^2Var(Y)\\ &= 9Var(X)+16Var(Y)\\ &= 9*2+16*3\\ &= 66\end{aligned}$$

14. The joint pdf of (X,Y) is given by $e^{-(x+y)}, 0<x,y<\infty$. Are X and Y independent?

Ans :

$$f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$=\int_0^\infty e^{-x}e^{-y}dy$$

$$=e^{-x}\left(-e^{-y}\right)_0^\infty=-e^{-x}(0-1)=e^{-x}.$$

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$$

$$=\int_0^\infty e^{-x}e^{-y}dx$$

$$=e^{-y}\left(-e^{-x}\right)_0^\infty=-e^{-y}(0-1)=e^{-y}.$$

$f(x)\,f(y)=e^{-x}e^{-y}=e^{-(x+y)}=f(x,y)$ Therefore, X any Y are independent.

15. The two lines of regression are $8x-10y+66=0$, $40x-18y-214=0$. Find the mean value of X and Y.

Ans:

$8x-10y=-66$ _________ (1)

$40x-18y=214$ ________(2)

Solving (1) and (2) , we get $x=104, y=17$

Mean of X = 13

Mean of Y = 17.

16. The two regression lines are $x=\frac{9}{20}y+\frac{107}{20}$, $y=\frac{4}{5}x+\frac{33}{5}$. Find correlation coefficient?

Ans:

$\mathbf{r = \sqrt{b_{xy}.b_{yx}}}$

Here, $\mathbf{b_{xy} = \frac{9}{20}}$, $\mathbf{b_{yx} = \frac{4}{5}}$

$$r=\sqrt{\frac{9}{20} X \frac{4}{5}}=0.6$$