

UNIT – 1 PROBABILITY AND RANDOM VARIABLES Two marks

1. Define i) Discrete random variable

ii) Continuous random variable

- i) Let X be a random variable, if the number of possible values of X is finite or count ably finite, then X is called a discrete random variable.
- ii) A random variable X is called the continuous random variable, if x takes all its possible values in an interval.
- 2. Define probability mass function (PMF):

Let X be the discrete random variable taking the values X_1 , X_2

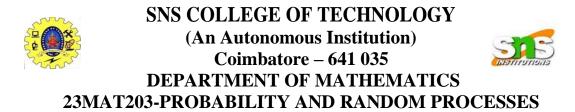
Then the number $P(X_i) = P(X = X_i)$ is called the probability mass function of X and it satisfies the following conditions.

- i) $P(X_i) \ge 0$ for all;
- ii) $\sum_{i=1}^{\infty} P(X_i) = 1$
- 3. Define probability Density function (PDF):

Let x be a continuous random variable. The Function f(x) is called the probability density function (PDF) of the random variable x if it satisfies.

- i) $f(x) \ge 0$
- ii) $\int_{-\infty}^{\infty} f(x) dx = 1$
- 4. Define cumulative distribution function (CDF):

Let x be a random variable. The cumulative distribution function, denoted by F(X) and is given by $F(X)=P(X \le x)$



5. If x is a discrete R.V having the p.m.f

X:	-1	0	1
P(X):	K	2k	3k

Find $P(x \ge 0)$

Answer:
$$6k = 1 \Rightarrow k = \frac{1}{6}$$

$$P[x \ge 0] = 2k + 3k \Rightarrow P[x \ge 0] = \frac{1}{6}$$

6. The random variable x has the p.m.f. P (x)= $\frac{x}{15}$, x=1,2,3,4,5 and = 0 else where.

- Find P $[\frac{1}{2} < x < \frac{5}{2}/x > 1]$.**Answer:** P $[\frac{1}{2} < x < \frac{5}{2}/x > 1] = \frac{P[x=2]}{P(x>1)} = \frac{P[x=2]}{1 - P(x \le 1)} = \frac{2/15}{1 - 1/15} = \frac{1}{7}$
- 7. If the probability distribution of X is given as :

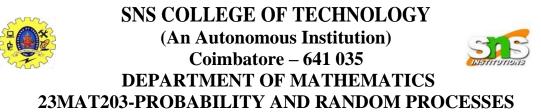
	Х	1	2	3	4
	P(X)	0.4	0.3	0.2	0.1
Find P $\left[\frac{1}{2} < x < \frac{7}{2}/x > 1\right]$.					

Answer :

$$P\left[\frac{1}{2} < x < \frac{7}{2}/x > 1\right] = \frac{P[1 < x < 7/2]}{P(x > 1)} = \frac{P(x = 2) + P(x = 3)}{1 - P(x = 1)} = \frac{0.5}{0.6} = \frac{5}{6}$$

8. A.R.V. X has the probability function

X	-2	-1	0	1
P(X)	0.4	k	0.2	0.3



Find k and the mean value of X

Answer:

k=0.1 Mean =
$$\sum xP(x) = \frac{1}{10} [-8-1+0+3] = -0.6$$

9. If the p.d.fof a R.V. X is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$, find

P[x > 1.5/x > 1].

Answer :

$$P[x > 1.5/x > 1] = \frac{P[x > 1.5]}{P(x > 1)} = \frac{\int_{1.5}^{2} \frac{x}{2} dx}{\int_{1}^{2} \frac{x}{2} dx} = \frac{4 - 2.25}{4 - 1} = 0.5833$$

10.If the p.d.f of a R.V.X is given by $f(x) = \{1/4, -2 < x < 2.0, \text{ else where. Find } P[|X|>1].$

Answer:

$$P[|X|>1] = 1 - P[|X|<1] = 1 - \int_{-1}^{1} \frac{1}{4} dx = \frac{1}{2}$$

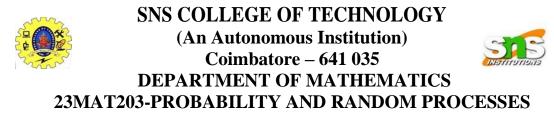
11. If $f(x) = kx^2$, 0<x<3 is to be density function, Find the value of k.

Answer:

$$\int_0^3 kx^2 dx = 1 \Rightarrow 9k = 1 \therefore k = \frac{1}{9}$$

12. If the c.d.f. of a R.V X is given by F(x) = 0 for x < 0; $= \frac{x^2}{16}$ for $0 \le x < 4$ and =

1 for
$$x \ge 4$$
, find $P(X > 1/X < 3)$.



$$P(X > 1/X < 3) = \frac{P[1 < X < 3]}{P[0 < X < 3]} = \frac{F(3) - F(1)}{F(3) - F(0)} = \frac{8/16}{9/16} = \frac{8}{9}$$

13. The cumulative distribution of X is $F(x) = \frac{x^3+1}{9}$, -1, < X < 2 and =

0, otherwise. Find P[0 <X<1].

Answer:

$$P[0 < X < 1] = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

14. A Continuous R.V X that can assume any value between x=2 and x=5 had the p.d.f f(x) = k(1+x). Find P(x<4).

Answer:

$$\int_{2}^{3} k(1+x)dx = 1 \Rightarrow \frac{27k}{2} = 1 \therefore k = \frac{2}{27}$$

$$P[X<4] = \int_{2}^{4} \frac{2}{27} (1+x)dx = \frac{16}{27}$$

15. The c.d.f of X is given by F (x) = $\begin{bmatrix} 0, x > 0 \\ x^2, & 0 \le x \le 1 \text{ Find the p.d.f of x, and} \\ 1, x > 1 \end{bmatrix}$

obtain P(X>0.75).

Answer:

$$F(x) = \frac{d}{dx}F(x) = \begin{bmatrix} 2x, 0 \le x \le 1\\ 0, otherwise \end{bmatrix}$$

 $P[x<0.75] = 1 - P[X \le 0.75] = 1 - F(0.75) = 1 - (0.75)^2 = 0.4375$

16. Check whether $f(x) = \frac{1}{4} x e^{-x/2}$ for $0 < x < \infty$ can be the p.d.f of X.



Answer:

$$= \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{x}{4} e^{-x/2} dx = \int_{0}^{\infty} t e^{-1} dt \text{ where } t = \frac{x}{2}$$
$$= (-te^{-1} - e^{-1})_{0}^{\infty} = -[0-1] = 1$$
$$\therefore f(x) \text{ is the } p.d.f \text{ of } X.$$

17.A continuous R.V X has a p.d.f $f(x) = 3x^2$, $0 \le x \le 1$. Find b such that

P(X > b) = 0.05.

Answer:

$$3\int_{b}^{1} x^{2} dx = 0.05 \Rightarrow 1 - b^{3} = 0.05 \Rightarrow b^{3} = 0.95 \therefore b = (0.95)^{\frac{1}{3}}$$

18.Let X be a random variable taking values -1, 0 and 1 such that P(X=-1) =

2P(X=0) = P(X=1). Find the mean of 2X-5.

Answer:

$$\sum P(X = x) = 1 \implies 5P(X = 0) = 1 : P(X = 0) = \frac{1}{5}$$

Probability distribution of X:

19. Find the cumulative distribution function F(x) corresponding to the p.d.f.



$$F(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty.$$

Answer

F(x) =
$$\int_{-\infty}^{x} f(x) dx = \frac{1}{\pi} \int_{-\infty}^{x} \frac{dx}{1+x^2} = \frac{1}{\pi} [tan^{-1}x]$$

= $\frac{1}{\pi} [\frac{\pi}{2} + tan^{-1}x]$

20. The diameter of an electric cable, say X is assumed to a continues R.V with

p.d.f of given by $f(x) = kx(1-x), 0 \le x \le 1$. Determine k and $P\left(x \le \frac{1}{3}\right)$

Answer:

$$\int_{0}^{1} kx(1-x)dx = 1 \implies k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \quad \therefore k = 6$$

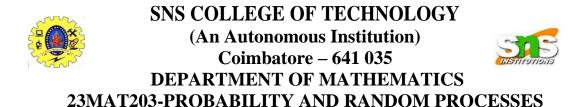
$$P\left[X \le \frac{1}{3}\right] = 6\int_0^{1/3} (x - x^2) dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1/3} = \left[(3x^2 - 2x^3)\right]_0^{1/3} = \frac{1}{3} - \frac{2}{27} = \frac{7}{27}$$

21. A random variable X has the p.d.f f(x) given by $f(x) = \begin{cases} Cxe^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$. Find the value of C and C.D.F of X.

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$$C\int_{0}^{\infty} xe^{-x}dx = 1 \Rightarrow C[x(-e^{-x}]_{0}^{\infty} = 1$$
$$\therefore C[-0+1] = 1 \Rightarrow C = 1$$

C.D.F:
$$F(x) = \int_0^x f(x) dx = 1 - (1+x)e^{-x}$$
 for $x \ge 0$.



22. State the properties of cumulative distribution function.

Answer:

- i) $F(-\infty)=0$ and $F(\infty) = 1$.
- ii) $F(\infty)$ is non decreasing function of X.
- iii) If $F(\infty)$ is the p.d. f of X, then f(x)=F'(x)
- iv) $P[a \le X \le b] = F(b) F(a)$
- 23. Define the raw and central moments of R.Vand state the relation between them.

Answer:

Raw moment $\mu'_r = E[X^r]$

Central moment $\mu_r = E[\{X - E(X)\}^r].$

$$\mu_r = \mu'_r - rC_1 \mu'_{r-1} \mu'_r + rC_2 \mu'_{r-2} (\mu'_r)^2 - \dots + (-1)^r (\mu'_1)^r$$

24. The first three moments of a R.V.X about 2 are 1, 16, -40. Find the mean, variance of X. Hence find μ_3 .

Answer:

$$E(X) = {\mu'}_1 + A \Rightarrow Mean = 1 + 2 = 3$$

Variance = $E(X^2) - [E((X)]^2 = 16 - 1 = 15$
 $\mu_3 = {\mu'}_3 - 3{\mu'}_2{\mu'}_1 + 2({\mu'}_1)^3 = -86$

25. Find the r-th moment about origin of the R.V X with p.d.ff(x) =

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\begin{bmatrix} Ce^{-ax}, x \ge 0\\ 0, else where \end{bmatrix}
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$$\int_{0}^{\infty} C e^{-ax} dx = 1 \Rightarrow C = a$$
$$\mu'_{r} = \int_{0}^{\infty} x^{r} f(x) dx = a \int_{0}^{\infty} x^{(r+1)-1} e^{-ax} dx = \frac{\sqrt{(r+1)}}{a^{r}} = \frac{r!}{a^{r}}$$

26. A C.R.V X has the p.d.f $f(x)=kx^2e^{-x}$, x > 0. Find the r-th moment about the origin.

Answer:

$$\int_{0}^{\infty} kx^{2}e^{-x}dx = 1 \Rightarrow k = \frac{1}{2}$$
$$\mu'_{1} = E[X^{r}] = \frac{1}{2}\int_{0}^{\infty} x^{r+2}e^{-x}dx = \frac{1}{2}\sqrt{(r+3)} = \frac{(r+2)!}{2}$$

27. If X and Y are independent R,V's and Z = X+Y, prove that $M_x(t)M_y(t)$.

Answer:

$$M_{z}(t) = E[e^{tz}] = E[e^{t(X+Y)}] = E[e^{tx}]E[e^{ty}]$$
$$= M_{x}(t)M_{y}(t).$$

28. If the MGF of X is $M_x(t)$ and if Y=aX+b show that $M_y(t) = e^{bt}M_x(at)$.

Answer:

$$M_{y}(t) = E[e^{ty}] = E[e^{bt}e^{axt}] = e^{bt}E[e^{(at)X}] = e^{bt}M_{x}(at).$$

29. If a R.V X has the MGF M(t)= $\frac{3}{3-t}$, obtain the mean and variance of X.

$$\mathbf{M}(t) = \frac{3}{3[1 - \frac{t}{3}]} = 1 + \frac{t}{3} + \frac{t^2}{9} + \dots$$

$$E(x) = \text{Co-efficient of } \frac{t}{1!} \text{in } (1) = \frac{1}{3}$$
$$E(X^2) = \text{co-efficient of } \frac{t^2}{2!} \text{in } (1) = \frac{1}{9}$$
$$\therefore \text{ Mean} = \frac{1}{3} \text{ and } V(X) = E(X^2) - [E(X)]^2 = \frac{1}{9}$$

30.If the r-th moment of a C.R.V X about the origin is r!, find the M.G. F of X.

Answer:

$$M_x(t) = \sum_{r=0}^{\infty} E[X^r] \cdot \frac{t^r}{r!} = \sum_{r=0}^{\infty} t^r$$
$$= 1 + t + t^2 + \cdots = (1 - t)^{-1} = \frac{1}{1 - t}$$

31. If the MGF of a R.V. X is $\frac{2}{2-t}$, Find the standard deviation of x.

Answer:

$$M_{x}(t) = \frac{2}{2-t} = (1 - \frac{t}{2})^{-1} = 1 + \frac{t}{2} + \frac{t^{2}}{4} + \cdots$$
$$E(X) = \frac{1}{2}; E(x^{2}) = \frac{1}{2}; V(X) = \frac{1}{4} \Rightarrow S.D \text{ of } X = \frac{1}{2}$$

32.Find the M.G.F of the R.V X having p.d.f $f(x) = \begin{bmatrix} \frac{1}{3}, -1 < x < 2\\ 0, else where \end{bmatrix}$

$$M_{x}(t) = \int_{-1}^{2} \frac{1}{3} e^{tx} dx = \frac{1}{3t} [e^{2t} - e^{-t}] \text{ for } t \neq 0$$

When t=0, $M_{x}(t) = \int_{-1}^{2} \frac{1}{3} dx = 1$
 $\therefore M_{x}(t) = \begin{bmatrix} \frac{e^{2t} - e^{-t}}{3t}, t \neq 0\\ 1, t = 0 \end{bmatrix}$

33. Find the MGF of a R.V X whose moments are given by $\mu'_r = (r = 1)!$

Answer:

$$M_x(t) = \sum_{r=0}^{\infty} E[X^r] \cdot \frac{t^r}{r!} = \sum_{r=0}^{\infty} (r+1)t^r$$
$$= 1 + 2t + 3t^2 + \dots = (1-t)^{-2}$$
$$\therefore M_x(t) = \frac{1}{(1-t)^2}$$

34. Give an example to show that if p.d.f exists but M.G.F. does not exist.

Answer:

$$P(x) = \begin{bmatrix} \frac{6}{\pi^2 x^2}, x = 1, 2, \dots \\ 0, otherwise \end{bmatrix}$$
$$\sum P(x) = \frac{6}{\pi^2} \Rightarrow \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{6}{\pi^2} \left[\frac{\pi^2}{6} \right] = 1$$

 \therefore P(x) is a p.d.f.

But $M_x(t) = \frac{6}{\pi^2} \sum \frac{e^{tx}}{x^2}$, which is a divergent series $\therefore M_x(t)$ doesnt exist.

35. The moment generating function of a random variable X is given by $M_x(t) =$

$$\frac{1}{3}e^{t} + \frac{4}{15}e^{3t} + \frac{2}{15}e^{4t} + \frac{4}{15}e^{5t}$$
. Find the probability density function of X.

Х	1	2	3	4
P(X)	1/3	4/15	2/15	4/15



36.Let
$$M_x(t) \frac{1}{(1-t)}$$
, $t < 1$ be the M.G.F of a R.V X. Find the MGF of the RV

Y = 2X + 1.

Answer:

If Y =aX+b,
$$M_y(t) = e^{bt}M_x(at)$$
 \therefore $M_y(t) = \frac{e^t}{1-2t}$.

37.Suppose the MGF of a RV X is of the form $M_x(t) = (0.4e^t + 0.6)^8$.What is the MGF of the random variable Y=3X+2.

Answer:

$$M_{y}(t) = e^{2t} M_{x}(3t) = e^{2t} [(0.4)e^{3t} = 0.6)]^{8}$$

38. The moment generating function of a RV X is $\left[\frac{1}{5} + \frac{4e^t}{5}\right]^{15}$. Find the MGF of

Y = 2X + 3.

Answer:

If Y = 2X + 3, then $M_y(t) = e^{3t}M_x(2t)$.

$$\therefore M_{y}(t) = e^{3t} \left[\frac{1}{5} + \frac{4e^{t}}{5} \right]^{15}$$

39. If a random variable takes the values -1, 0 and 1 with equal probabilities, find the MGF of X.

$$M_x(t) = \sum e^{tx} P(x) = \frac{1}{3}e^{-1} + \frac{1}{3} + \frac{1}{3}e^{-1} = \frac{1}{3}[1 + e^{-1} + e^{-1}]$$