SNS COLLEGE OF TECHNOLOGY
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DEPARTMENT OF MATHEMATICS
23MAT203-PROBABILITY AND RANDOM PROCESSES

UNIT 4
CORRELATION AND SPECTRAL DENSITIES
Two marks

1. Define the ACF.

Answer:

Let $X(t_1)$ and $X(t_2)$ be two random variables. The autocorrelation of the random process $\{X(t)\}$ is

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)].$$

If $t_1 = t_2 = t$, $R_{XX}(t,t) = E[X^2(t)]$ is called as mean square value of the random process.

2. State any four properties of Autocorrelation function.

Answer:

1. $R_{XX}(-\tau) = R_{XX}(\tau)$
2. $|R(\tau)| \le R(0)$
3. $R(\tau)$ *is continuous for all* τ.
4. If $R(\tau)$ is ACF of a stationary RP $\{X(t)\}$ with no periodic components, then $\mu_X{}^2 = \lim_{\tau \to \infty} R(\tau)$.

3. Define the cross – correlation function.

Answer:

Let $\{X(t)\}$ and $\{Y(t)\}$ be two random processes. The cross-correlation is

$$R_{XY}(\tau) = E[X(t)Y(t-\tau)] .$$

4. State any two properties of cross-correlation function.

Answer:

1. $R_{YX}(-\tau) = R_{XY}(\tau)$
2. $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)R_{YY}(0)} \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

5. Given the ACF for a stationary process with no periodic component is $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$ .find the mean and variance of the process{X(t)}

Answer:

By the property of ACF

$$\mu_x^{\ 2} = \lim_{\tau\to\infty} R_{XX}(\tau) = \lim_{\tau\to\infty} 25 + \frac{4}{1+6\tau^2} = 25$$

$$\mu_x = 5$$

$E\{X^2(t)\}=R_{xx}(0)=25+4=29$

$Var\{X(t)\}=E\{X^2(t)\}-E^2\{X(t)\}=29-25=4.$

6. ACF: $R_{XX}(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$ .find mean and variance.

7. ACF: $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$ .find mean and variance.

8. Define power spectral density.

Answer:

If $R_{XX}(\tau)$ is the ACF of a WSS process {X(t)} then the power spectral density $S_{XX}(\omega)$ of the process {X(t)}, is defined by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau}\, d\tau \quad \text{(or)} \quad S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i2\pi f\tau}\, d\tau$$

9. Express each of ACF and PSD of a stationary R.P in terms of the other.{(or) write down wiener khinchine relation }

Answer:

$R_{XX}(\tau)$ and $S_{XX}(\omega)$ are Fourier transform pairs.

i.e., $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau}\, d\tau$ and $R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau}\, d\tau$

10. Define cross power spectral density of two random process {X(t)} and{Y(t)}.

Answer:

If {X(t)} and {Y(t)} are jointly stationary random processes with cross correlation function $R_{XY}(\tau)$,then cross power spectral density of {X(t)} and {Y(t)}is defined by

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau}\, d\tau$$

11. State any two properties of power spectral density.

Answer:

i) $S(\omega) = S(-\omega)$

ii) $S(\omega) > 0$

iii)The spectral density of a process {X(t)},real or complex, is a real function of ω and non-negative.

12. If $R(\tau) = e^{-2\lambda|\tau|}$ is the ACF of a R.P{X(t)}, obtain the spectral density.

Answer:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau}\, d\tau = \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} e^{-i\omega\tau}\, d\tau = 2\int_{0}^{\infty} e^{-2\lambda\tau} \cos\omega\tau\, d\tau = \frac{4\lambda}{4\lambda^2 + \omega^2}.$$

13. State any four properties of cross power density spectrum.

Answer:

i) $S_{XY}(\omega) = S_{YX}(-\omega) = S^*_{YX}(\omega)$

ii) Re[$S_{XY}(\omega)$] and Re[$S_{YX}(\omega)$] are even function of ω

iii) Im[$S_{XY}(\omega)$] and Im[$S_{YX}(\omega)$] are odd function of ω

iv) $S_{XY}(\omega)$ =0and $S_{YX}(\omega)$ =0 if X(t) and Y(t) are orthogonal.

Unit 5
LINEAR SYSTEMS WITH RANDOM INPUTS
Two marks

14. Define a system and Define the linear system.

Answer:

A system is a functional relationship between the input X(t) and the output Y(t).i.e., $Y(t) = f[X(t)]$, $-\infty < t < \infty$.

A System is a functional relationship between the input X(t) and the output Y(t).

If $f[a_1X_1(t)+ a_2X_2(t)] = a_1 f[X_1(t)]+ a_2 f[X_2(t)]$, then f is called a linear system.

15. Define time invariant system.

Answer:

If $Y(t+h) = f[X(t+h)]$ where $Y(t) = f[X(t)]$, then f is called the time invariant system.

16. Check whether the following system is linear .y(t)=t x(t)

Answer:

Consider two input functions $x_1(t)$ and $x_2(t)$. The corresponding outputs are

$y_1(t)=t\, x_1(t)$ and $y_2(t)=t\, x_2(t)$

Consider $y_3(t)$ as the linear combinations of the two inputs.

$y_3(t)= t[a_1 x_1(t)+a_2 x_2(t)]= a_1t\, x_1(t)+a_2\, t\, x_2(t)$ ……………..(1)

consider the linear combinations of the two outputs.

$a_1y(t)+a_2 y_2(t)= a_1t\, x_1(t)+a_2\, t\, x_2(t)$ …………………(2)

From (1)and(2), (1)=(2)

The system y(t)=t x(t) is linear.

17.

Check whether the following system is linear .y(t)= x²(t)

Answer:

Consider two input functions $x_1(t)$ and $x_2(t)$. The corresponding outputs are

$y_1(t)=x_1^2(t)$ and $y_2(t)=x_2^2(t)$

Consider $y_3(t)$ as the linear combinations of the two inputs.

$y_3(t)= [a_1 x_1(t)+a_2 x_2(t)]^2= a_1^2 x_1^2(t)+a_2^2 x_2^2(t)+2\, a_1x_1(t)a_2x_2(t)$ ……..(1)

consider the linear combinations of the two outputs.

$a_1y(t)+a_2 y_2(t)= a_1 x_1^2(t)+a_2 x_2^2(t)$ …………………(2)

From (1) and (2), $(1) \neq (2)$

The system $y(t)=x^2(t)$ is not linear.

18. **Define the Linear Time Invariant System.**

Answer:

A linear system is said to be also time-invariant if the form of its impulse response $h(t,u)$ does not depend on the time that the impulse is applied.

For linear time invariant system, $h(t,u) = h(t-u)$

If a system is such that its Input X(t) and its Output Y(t) are related by a Convolution integral,

i.e., if $Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$, then the system is a linear time-invariant system.

19. **Find the ACF of the random process {X(t)}, if its power spectral density is given by**

$$S(\omega) = \begin{cases} 1+\omega^2, & for\ |\omega| \le 1 \\ 0 & , for\ |\omega| > 1 \end{cases}$$

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1}\{1+\omega^2\}e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1}\{e^{i\omega\tau} + \omega^2 e^{i\omega\tau}\}d\omega = \frac{1}{2\pi}\left\{\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-1}^{1} + \int_{-1}^{1}\omega^2\cos\omega\tau\, d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-}^{1} + 2\int_{0}^{1}\omega^2\cos\omega\tau\, d\omega\right\} = \frac{1}{2\pi}\left[\frac{e^{i\tau}-e^{-i\tau}}{i\tau}\right] + 2\left[\omega^2\frac{\sin\omega\tau}{\tau} + \frac{2\omega\cos\omega\tau}{\tau^2} - \frac{2\sin\omega\tau}{\tau^3}\right]_0^1$$

$$= \frac{1}{2\pi}\left[\frac{2\sin\tau}{\tau} + \frac{2\sin\tau}{\tau} + \frac{4\cos\tau}{\tau^2} - \frac{4\sin\tau}{\tau^3}\right] = \frac{1}{2\pi}\left[\frac{2\tau^2\sin\tau + 2\tau^2\sin\tau + \tau 4\cos\tau - 4\sin\tau}{\tau^3}\right]$$

$$= \frac{2\{\tau^2\sin\tau + \tau\cos\tau - \sin\tau\}}{\pi\tau^3}$$

20. **Define Average power.**

$$\text{Average power } = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = R(0)$$

21. A system has an impulse response $h(t) = e^{-\beta t}U(t)$, find the system transfer function.

Solution:

The unit step function $U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

$$h(t) = \begin{cases} 0, & t < 0 \\ e^{-\beta t}, & t \geq 0 \end{cases}$$

$$\therefore H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\beta t}\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(\beta + i\omega)t}\, dt$$

$$= \left[\frac{e^{-(\beta+i\omega)t}}{-(\beta+i\omega)} \right]_0^{\infty}$$

$$= -\frac{1}{\beta+i\omega} \left[e^{-(\beta+i\omega)t} \right]_0^{\infty}$$

$$= -\frac{1}{\beta+i\omega} [0-1]$$

$$= \frac{1}{\beta+i\omega}$$

22. State any properties of Linear time invariant system.

If the input $x(t)$ and its output $y(t)$ are related by $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$, then the system is linear time invariant system