Linear Systems with random inputs

* Input - output relationship in the time domain

1. If h(t) is the impulse response of the linear system & y(t) is the output response of the system for the input x(t), then the output correlation function is given by,

$$R_{yy}(\tau) = h(-\tau) * h(\tau) * R_{XX}(\tau)$$

2. The cross correlation function between the input X(t) and the output Y(t) is given by,

$$R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$$

* Input - output relationship in the frequency domain

1. If h(t) is the unit impulse response of the linear system & y(t) is the response of the system for the input x(t), then

$$S_y(f) = |H(f)|^2 S_x(f)$$

2). The cross power density spectrum between the input and output processes of a linear system is gvn. by

$$S_{XY}(f) = H(f) \cdot S_X(f)$$

$\omega \to 2\pi f$

Mean and mean-square value of the input:

The mean of the output of a linear system is given by $\overline{Y} = H(0)\,\overline{X}$

1. Consider a linear System as shown below :

$$X(t) \longrightarrow \boxed{\dfrac{1}{6+j\omega}} \longrightarrow Y(t)$$

$X(t)$ is the input and $Y(t)$ is the output of the system. The auto correlation of $X(t)$ is $R_{XX}(\tau) = 3 \cdot \delta(\tau)$. Find the power spectral density, auto correlation function and mean square value of the output $Y(t)$.

Soln.

The input auto correlation is gvn. by,

$$R_{XX}(\tau) = 3 \cdot \delta(\tau)$$

$$\therefore \; S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} 3 \cdot \delta(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$S_{XX}(\omega) = 3 \qquad \because F[\delta(\tau)] = 1$$

and

$$H(\omega) = \frac{1}{6+j\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{6^2+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{36+\omega^2}$$

$$\therefore \; R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{3}{36+\omega^2}\, e^{i\omega\tau}\, d\omega$$

$$= \frac{3}{2\pi}\left[\frac{\pi}{6} e^{-6|\tau|}\right]$$

$$\left[\because \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\omega^2+a^2}\,d\omega = \frac{\pi}{a} e^{-a|\tau|}\right]$$

$$R_{yy}(\tau) = \frac{1}{4} e^{-6|\tau|}$$

$\therefore$ Mean square value of the output $\rightarrow R_{yy}(0)$

$$R_{yy}(0) = \frac{1}{4} e^{-6(0)}$$

$$= \frac{1}{4}$$

$Y = H(0) X$

3. Assume a random process $X(t)$ is given as input to a system with transfer function

$$H(f) = 1, \quad -W < f < W$$

Find the output correlation function and power of output process. Assume the autocorrelation of input process as $\frac{\eta_0}{2} \delta(\tau)$.

Soln.

Given $R_{XX}(\tau) = \frac{\eta_0}{2} \delta(\tau)$

Taking Fourier transform of $R_{XX}(\tau)$,

$$\therefore S_X(f) = \frac{\eta_0}{2}, \quad -\infty < f < \infty$$

$$\because F[\delta(\tau)] = 1$$

$$\therefore S_Y(f) = |H(f)|^2 S_X(f)$$

$$= 1\left(\frac{\eta_0}{2}\right)$$

$$= \frac{\eta_0}{2}, \quad -W \le f \le W$$

power of $Y(t) = \int_{-W}^{W} S_Y(f)\, df$

$$= 2\int_{0}^{\omega} \frac{\eta_0}{2}\, df$$

power of $Y(t) = \eta_0 \omega$

$$\therefore R_{YY}(\tau) = \int_{-\omega}^{\omega} \frac{\eta_0}{2} e^{i2\pi f\tau}\, df$$

$$= \frac{\eta_0}{2} \int_{-\omega}^{\omega} (\cos 2\pi f\tau + i \sin 2\pi f\tau)\, df$$

$$= \frac{\eta_0}{2}\left[2\int_{0}^{W} \cos 2\pi f\tau\, df + i(0)\right]$$

$$= \eta_0 \left[\frac{\sin 2\pi f\tau}{2\pi\tau}\right]_{0}^{\omega}$$

$$= \frac{\eta_0 W}{2\pi\tau W}\left[\sin 2\pi\omega\tau - 0\right]$$

$$R_{YY}(\tau) = \frac{\eta_0 W \sin(2\pi\omega\tau)}{2\pi\tau W} = \eta_0 W\left[\frac{\sin(2\pi W\tau)}{2\pi W\tau}\right], \quad -\infty < \tau < \infty$$

Power of $Y(t) = R_{YY}(0) = \eta_0 \omega \cdot 1$

$$= \eta_0 W. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$