Unit - 5

Linear Systems with Random Inputs

* Linear time invariant system
* System transfer function
* Linear system with random inputs
* Auto correlation and cross correlation functions of input and output.

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Linear time invariant system:

System:

A system is defined by a functional relationship between the input $x(t)$ and the output $y(t)$ as,

$$y(t) = f\{x(t)\}, \quad -\infty < t < \infty$$

Linear system:

A system with functional relationship $f\{x(t)\}$ is linear, if for any two inputs $x_1(t)$ and $x_2(t)$, the output of the system can be defined as $f\{(a_1 x_1(t) + a_2 x_2(t)\}$

$$= a_1 f\{x_1(t)\} + a_2 f\{x_2(t)\}$$

Time invariant system:

Let $Y(t) = f\{X(t)\}$.

If $Y(t+h) = f\{X(t+h)\}$, then f is called a time invariant system of $X(t)$ and $Y(t)$.

Causal system:

Suppose that the value of the output $Y(t)$ at $t = t_0$ depends only on the past values of the input $X(t)$ at $t = t_0$.

i.e., $Y(t_0) = f\{X(t) ; t \le t_0\}$

Stable :

A system is stable if for every bounded input, the system given bounded output.

Note :

* $Y(t) = h(t) * X(t)$

where $h(t)$ is weighting function (or) impulse response function and $X(t)$ is Input.

* $$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$$

(or)

$$Y(t) = \int_{-\infty}^{\infty} h(t-u)\, X(u)\, du$$

* $H(\omega) = F[h(\tau)] \rightarrow$ System transfer funct

* Unit impulse response

$$\delta(t-a) = \begin{cases} \frac{1}{\varepsilon}, & a - \frac{\varepsilon}{2} \le t \le a + \frac{\varepsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

where $\varepsilon \rightarrow 0$.

Properties of linear System:

Property 1:

If the input $x(t)$ and its output $y(t)$ are related by $y(t) = \int_{-\infty}^{\infty} b(u)\, x(t-u)\, du$, then the system is a linear time-invariant system.

Proof:

i). To prove $y(t)$ is linear.

Let $x(t) = a_1 x_1(t) + a_2 x_2(t)$

Then
$$y(t) = \int_{-\infty}^{\infty} b(u)\, x(t-u)\, du$$
$$= \int_{-\infty}^{\infty} b(u)\left[a_1 x_1(t-u) + a_2 x_2(t-u)\right] du$$
$$= \int_{-\infty}^{\infty} b(u)\, a_1 x_1(t-u)\, du + \int_{-\infty}^{\infty} b(u)\, a_2 x_2(t-u)\, du$$
$$= a_1 \int_{-\infty}^{\infty} b(u)\, x_1(t-u)\, du + a_2 \int_{-\infty}^{\infty} b(u)\, x_2(t-u)\, du$$
$$y(t) = a_1 y_1(t) + a_2 y_2(t)$$

$\therefore$ $y(t)$ is a linear

ii). To prove $y(t)$ is time invariant.

Let $y(t) = \int_{-\infty}^{\infty} b(u)\, x(t-u)\, du$

If $x(t)$ is replaced by $x(t+h)$, then
$$y(t) = \int_{-\infty}^{\infty} b(u)\, x(t+h-u)\, du$$

$$y(t) = y(t+h)$$

$\therefore$ $y(t)$ is time invariant.

Hence the system is linear time invariant system

Property 2:

If $\{x(t)\}$ is a WSS process and if $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$, then we've the following results.

a). $R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$

b). $R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$

c). $S_{xy}(\omega) = S_{xx}(\omega) * H(\omega)$

d). $S_{yy}(\omega) = S_{xx}(\omega)\, |H(\omega)|^2$

Proof:

Given $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$

a). $R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$

$$= E\left[x(t) \int_{-\infty}^{\infty} h(u)\, x(t+\tau-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} h(u)\, E[x(t)\, x(t+\tau-u)]\, du$$

$$= \int_{-\infty}^{\infty} h(u)\, R_{xx}(\tau-u)\, du \quad \{\because x(t) \text{ is WSS}\}$$

$R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$ (By convolution) b). $R_{yy}(\tau) = E[y(t)\, y(t+\tau)]$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, x(t-u)\, du \;\; y(t+\tau)\right]$$

$$= \int_{-\infty}^{\infty} h(u)\, E[x(t-u)\, y(t+\tau)]\, du$$

Put $t-u = t_1$, $t = t_1 + u$ | $t+\tau = t_1 + u + \tau$

$$= \int_{-\infty}^{\infty} h(u)\, E[x(t_1)\, y(t_1+u+\tau)]\, du$$

$$= \int_{-\infty}^{\infty} h(u)\, E[x(t_1)\, y(t_1+\tau+u)]\, du$$

$$= \int_{-\infty}^{\infty} h(u)\, R_{xy}(\tau+u)\, du$$

Take $u = -\alpha$
$du = -d\alpha$
$u = \infty \Rightarrow \alpha = -\infty$
$u = -\infty \Rightarrow \alpha = \infty$

$$\therefore R_{yy}(\tau) = \int_{\infty}^{-\infty} h(-\alpha)\, R_{xy}(\tau-\alpha)\, (-d\alpha)$$

$$= -\int_{\infty}^{-\infty} R_{xy}(\tau-\alpha)\, h(-\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau-\alpha)\, h(-\alpha)\, d\alpha$$

$$R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$$

$$\because y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du = x(t) * h(t)$$

c). WKT $R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$ by (a)

Taking Fourier Transform on bothsides,

$$F[R_{xy}(\tau)] = F[R_{xx}(\tau) * h(\tau)]$$

$$= F[R_{xx}(\tau)] * F[h(\tau)]$$

$$S_{xy}(\omega) = S_{xx}(\omega) * H(\omega) \quad \text{where } H(\omega) = F[h$$

d). WKT $R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$ by (b)

Taking FT on bothsides,

$$F[R_{yy}(\tau)] = F[R_{xy}(\tau) * h(-\tau)]$$

$$= F[R_{xy}(\tau)] * F[h(-\tau)]$$

$$S_{yy}(\omega) = S_{xy}(\omega) * H^*(\omega) \quad \left[\because H(\omega) = F[h(\tau)]\right.$$

$$H^*(\omega) = F[h(-\tau)]$$

$$= S_{xx}(\omega) * H(\omega) * H^*(\omega)$$

from (c)

$$\Rightarrow S_{yy}(\omega) = S_{xx}(\omega) * |H(\omega)|^2$$

Hence proved.

property 3 :

If the input to a time invariant, stable linear system is a WSS process, then the output will also be a WSS process.

(or)

To show that if the input $\{X(t)\}$ is a WSS process, then the output $\{Y(t)\}$ is a WSS process.

Proof :

i). WKT, the input and the output are related by,

$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du \longrightarrow (1)$$

i). $E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(u)\; X(t-u)\, du\right]$

$$= \int_{-\infty}^{\infty} h(u)\, E[X(t-u)]\, du$$

$X(t)$ is WSS

$\Rightarrow E[X(t)] = \text{Constant}$

$\therefore E[X(t-u)] = \text{Constant}$

= a finite constant,

[$\because$ Independent of t
System is stable]

= a constant

ii). $R_{YY}(t, t+\tau) = E[Y(t)\; Y(t+\tau)]$

$$= E\left[\int_{-\infty}^{\infty} h(u_1)\, X(t-u_1)\, du_1 \int_{-\infty}^{\infty} h(u_2)\, X(t+\tau-u_2)\, du_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, E\left[X(t-u_1)\, X(t+\tau-u_2)\right] du_1\, du_2 \rightarrow (1)$$

Since, $\{X(t)\}$ is a WSS process

$\Rightarrow E[X(t-u_1)\, X(t+\tau-u_2)]$ is a function of τ, Say $g(\tau)$

$$(1) \Rightarrow \quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, g(\tau)\, du_1\, du_2$$

$$= g(\tau) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, du_1\, du_2$$

= a function of τ

i.e., $R_{yy}(t, t+\tau) = R_{yy}(\tau)$

$\therefore$ $\{Y(t)\}$ is a WSS Process.