6]. If $X(t)$ is the input voltage to a circuit and $Y(t)$ is the output voltage. $\{X(t)\}$ is a stationary r.p. with $\mu_X = 0$ and $R_{XX}(\tau) = e^{-2|\tau|}$. Find the mean μ_Y and power spectrum $S_{YY}(\omega)$ of the output if the system transfer function is given by $H(\omega) = \frac{1}{\omega + 2i}$

Soln.

Given $\mu_X = 0$, $R_{XX}(\tau) = e^{-2|\tau|}$

WKT
$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$$

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} h(u)\, E[x(t-u)]\, du$$

$$\mu_y = 0 \qquad \because \mu_x = 0$$

Input of PSD:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} (\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} \cos\omega\tau\, d\tau - i\int_{-\infty}^{\infty} e^{-2|\tau|} \sin\omega\tau\, d\tau$$

$$= 2\int_{0}^{\infty} e^{-2\tau} \cos\omega\tau\, d\tau - i(0)$$

$$= 2\, \frac{2}{2^2+\omega^2}$$

$$S_{XX}(\omega) = \frac{4}{4+\omega^2}$$

and $$H(\omega) = \frac{1}{\omega+2i} = \frac{1}{\omega+2i}\, \frac{\omega-2i}{\omega-2i} = \frac{\omega-2i}{4+\omega^2}$$

$$|H(\omega)| = \sqrt{\left(\frac{\omega}{\omega^2+4}\right)^2 + \left(\frac{-2}{\omega^2+4}\right)^2} = \sqrt{\frac{\omega^2+4}{(\omega^2+4)^2}}$$

$$\Rightarrow |H(\omega)| = \frac{1}{\sqrt{\omega^2+4}}$$

$$\Rightarrow |H(\omega)|^2 = \frac{1}{\omega^2+4}$$

$$(1) \Rightarrow \quad S_{yy}(\omega) = \left(\frac{4}{4+\omega^2}\right)\left(\frac{1}{\omega^2+4}\right)$$

$$= \frac{4}{(\omega^2+4)^2}$$

7. A random process $x(t)$ is the input to a linear system whose impulse function is $h(t) = 2e^{-t}$, $t \geq 0$. The auto correlation function of the process is $R_{xx}(\tau) = e^{-2|\tau|}$. Find PSD of output process $y(t)$.

Soln.

Given $h(t) = 2e^{-t}$, $t \geq 0$

WKT

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} 2e^{-t}\, e^{-i\omega t}\, dt$$

$$= 2\int_{0}^{\infty} e^{-(1+i\omega)t}\, dt$$

$$= 2\left(\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right)_0^{\infty}$$

$$= \frac{-2}{1+i\omega}(0-1)$$

$$H(\omega) = \frac{2}{1+i\omega}$$

$$|H(\omega)|^2 = \frac{4}{1+\omega^2}$$

Input of PSD:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} (\cos\omega\tau - i\sin\omega\tau) d\tau$$

$$= 2\int_{0}^{\infty} e^{-2\tau} \cos\omega\tau \, d\tau - i(0)$$

$$= 2 \frac{2}{4+\omega^2} \qquad \because \int_0^\infty e^{-ax}\cos bx \, dx = \frac{a}{a^2+b^2}$$

$$= \frac{4}{4+\omega^2}$$

Output of PSD:

$$S_{YY}(\omega) = S_{XX}(\omega) * |H(\omega)|^2$$

$$= \frac{4}{4+\omega^2} \frac{4}{1+\omega^2}$$

$$= \frac{16}{(1+\omega^2)(4+\omega^2)}$$

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8). A WSS noise process $N(t)$ has an auto correlation function $R_{NN}(\tau) = Pe^{-3|\tau|}$, where P is a constant. Find its power spectrum.

Soln.:

Given $R_{NN}(\tau) = Pe^{-3|\tau|}$

$$S_{NN}(\omega) = \int_{-\infty}^{\infty} Pe^{-3|\tau|} e^{-i\omega\tau} d\tau$$

$$= P\int_{-\infty}^{\infty} e^{-3|\tau|} (\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$$= P\, 2\int_{0}^{\infty} e^{-3\tau} \cos\omega\tau \; d\tau - i(0)$$

$$= 2P \frac{3}{3^2+\omega^2}$$

$$= \frac{6P}{9+\omega^2}$$