



Linear Systems with random inputs

* Input-output relationship in the time domain

1. If $h(t)$ is the impulse response of the linear system & $y(t)$ is the output response of the system for the input $x(t)$, then the output correlation function is given by,

$$R_{yy}(\tau) = h(-\tau) * h(\tau) * R_{xx}(\tau)$$

2. The cross correlation function between the input $x(t)$ and the output $y(t)$ is given by,

$$R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$$

* Input-output relationship in the frequency domain

1. If $h(t)$ is the unit impulse response of the linear system & $y(t)$ is the response of the system for the input $x(t)$, then

$$S_y(f) = |H(f)|^2 S_x(f)$$



2]. The cross power density spectrum between the input and output processes of a linear system is given by

$$S_{xy}(f) = H(f) \cdot S_x(f)$$

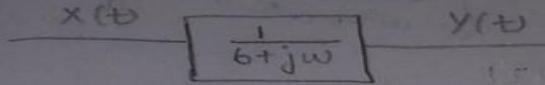
$\omega \rightarrow 2\pi f$

mean and mean-square value of the input:

The mean of the output of a linear system is given by $\bar{y} = H(0) \bar{x}$

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7. Consider a linear system, as shown below :



$X(t)$ is the input and $Y(t)$ is the output of the system. The auto correlation of $X(t)$ is $R_{XX}(\tau) = 3 \cdot \delta(\tau)$. Find the power spectral density, auto correlation function and mean square value of the output $Y(t)$.

Soln.

The input auto correlation is given by,

$$R_{XX}(\tau) = 3 \cdot \delta(\tau)$$

$$\therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} 3 \cdot \delta(\tau) e^{-i\omega\tau} d\tau$$

$$S_{XX}(\omega) = 3$$

$$\therefore F[\delta(\tau)] = 1$$

and $H(\omega) = \frac{1}{6+j\omega}$

$$|H(\omega)| = \frac{1}{\sqrt{6^2 + \omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{36 + \omega^2}$$

$$\therefore R_{YY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{3}{36 + \omega^2} e^{i\omega\tau} d\omega$$



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$$= \frac{3}{2\pi} \left[\frac{\pi}{6} e^{-6|\tau|} \right]$$
$$\left[\because \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\omega^2 + a^2} d\omega = \frac{\pi}{a} e^{-a|\tau|} \right]$$
$$R_{yy}(\tau) = \frac{1}{4} e^{-6|\tau|}$$

\therefore Mean square value of the output $\rightarrow R_{yy}(0)$

$$R_{yy}(0) = \frac{1}{4} e^{-6(0)}$$
$$= \frac{1}{4}$$



Q. Assume a random process $X(t)$ is given as input to a system with transfer function

$$H(f) = 1, \quad -W < f < W$$

Find the output correlation function and power of output process. Assume the autocorrelation of input process as $\frac{\eta_0}{2} \delta(\tau)$.

Soln.

Given $R_{XX}(\tau) = \frac{\eta_0}{2} \delta(\tau)$

Taking Fourier transform of $R_{XX}(\tau)$,

$$S_X(f) = \frac{\eta_0}{2}, \quad -\infty < f < \infty$$

$$\therefore F[\delta(\tau)] = 1$$

$$\therefore S_Y(f) = |H(f)|^2 S_X(f)$$

$$= 1 \left(\frac{\eta_0}{2} \right)$$

$$= \frac{\eta_0}{2}, \quad -W \leq f \leq W$$

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$$\text{power of } y(t) = \int_{-W}^W S_y(f) df$$

$$= 2 \int_0^W \frac{\eta_0}{2} df$$

$$\text{power of } y(t) = \eta_0 W$$

$$\therefore R_{yy}(\tau) = \int_{-W}^W \frac{\eta_0}{2} e^{i2\pi f\tau} df$$

$$= \frac{\eta_0}{2} \int_{-W}^W (\cos 2\pi f\tau + i \sin 2\pi f\tau) df$$

$$= \frac{\eta_0}{2} \left[2 \int_0^W \cos 2\pi f\tau df + i(0) \right]$$

$$= \eta_0 \left[\frac{\sin 2\pi f\tau}{2\pi\tau} \right]_0^W$$

$$= \frac{\eta_0 W}{2\pi\tau W} [\sin 2\pi W\tau - 0]$$

$$R_{yy}(\tau) = \frac{\eta_0 W \sin(2\pi W\tau)}{2\pi\tau W} = \eta_0 W \left[\frac{\sin(2\pi W\tau)}{2\pi W\tau} \right]$$

$$\text{power of } y(t) = R_{yy}(0) = \eta_0 W \cdot 1$$

$$= \eta_0 W.$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$