Harmonic functions :

An expression of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called the Laplace equation in two dimension.

Any function having continuous second order partial derivatives which satisfies the Laplace equation is called harmonic function.

Any two harmonic functions u and v such that $f(z) = u + iv$ is analytic are called Conjugate harmonic functions.

Note:

Both real and imaginary parts of an analytic function are harmonic. But the converse need not be true.

(1) Give an example such that u and v are harmonic but $u+iv$ is not analytic.

Soln:

Let $w = \bar{z}$

$u+iv = x - iy$

$\Rightarrow u = x \; ; \; v = -y$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0 \; ; \; \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0 \; ; \; \frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\Rightarrow u$ and v are harmonic.

But $u_x \neq v_y$ and $u_y = -v_x$

$\therefore f(z) = u+iv$ is not analytic.

(2) Prove that $u = e^x \cos y$ is a harmonic function.

Soln:

Let $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \; ; \; \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \; ; \; \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore u$ is a harmonic function.

(3) Prove that $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$ are harmonic but $u+iv$ is not a regular function.

Soln:

Let $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

$\Rightarrow u$ is a harmonic function.

Let $v = \frac{-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{-[(x^2+y^2)0 - y(2x)]}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2 2y - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-[(x^2+y^2) - y \cdot 2y]}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2 2y - (y^2-x^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

PROBLEMS:

(1) Prove that $w = z^2$ is analytic.

Soln:

We know $z = x + iy$

$\therefore \; w = z^2$

$= (x+iy)^2$

$= x^2 - y^2 + 2ixy$

$u + iv = (x^2 - y^2) + i(2xy)$

$u = x^2 - y^2 \quad ; \quad v = 2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x$

$u_x = v_y \;\; \& \;\; u_y = -v_x$

It satisfies CR equations.

$\Rightarrow w = z^2$ is analytic.

(2) Determine whether the function $w = 2xy + i(x^2 - y^2)$ is analytic.

Soln: $w = 2xy + i(x^2 - y^2)$

$u + iv = 2xy + i(x^2 - y^2)$

$u = 2xy \;,\; v = x^2 - y^2$

$u_x = 2y \;,\; v_x = 2x$

$u_y = 2x \qquad v_y = -2y$

$u_x \neq v_y$

$\& \; u_y \neq -v_x$

It doesn't satisfies CR equations

$\Rightarrow w = 2xy + i(x^2 - y^2)$ is not analytic.

$$= \frac{(x^2+y^2)^2 2y - 4y(y^2-x^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)2y - 4y(y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\boxed{\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3} + \frac{6x^2y - 2y^3}{(x^2+y^2)^3} = 0.$$

$\Rightarrow v$ is a harmonic function.

But $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$f(z) = u+iv$ is <u>not analytic</u> (or) <u>not regular</u> function.