



## DEPARTMENT OF MATHEMATICS

### Construction of Conjugate harmonic fns :

#### Method 1 :

Suppose  $u$  is given, then

$$v = \int \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c, \text{ where } c \text{ is a}$$

Constant.

#### Method 2 :

Suppose  $v$  is given, then

$$u = \int \left( \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \right) + c, \text{ where } c \text{ is a}$$

Constant.

- ① Show that the function  $u = \frac{1}{2} \log(x^2 + y^2)$  is harmonic and find its harmonic conjugate.

Soln :

$$\text{Let } u = \frac{1}{2} \log(x^2 + y^2)$$

$$u_x = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2}$$

$$u_{xx} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)

COIMBATORE-641 035, TAMIL NADU



## DEPARTMENT OF MATHEMATICS

$$u_y = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2y = \frac{y}{x^2+y^2}$$

$$u_{yy} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore u_{xx} + u_{yy} = 0$$

$u$  satisfies Laplace equation

$u$  is harmonic.

$$v = \int \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

$$= \int \left( \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \right) + c$$

$$= \int \frac{x dy - y dx}{x^2+y^2} + c$$

$$= \int \frac{x dy - y dx}{x^2(1+y^2/x^2)}$$

$$= \int \frac{d(y/x)}{1+y^2/x^2} + c$$

$$v = \tan^{-1}(y/x) + c$$

- (2) Prove that the function  $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$  is harmonic find the conjugate harmonic  $f_v$

Soln:

$$\text{Let } u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$u_x = 3x^2 - 3y^2 + 6x \quad ; \quad u_y = -6xy - 6y$$

$$u_{xx} = 6x + 6 \quad ; \quad u_{yy} = -6x - 6$$

$$u_{xx} + u_{yy} = (6x+6) + (-6x-6) = 0$$

**DEPARTMENT OF MATHEMATICS**

(3) prove that  $u = x^2 - y^2$ ,  $v = \frac{-y}{x^2 + y^2}$  are harmonic

but  $u + iv$  is not a regular function.

Soln:

$$\text{Let } u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\Rightarrow u$  is a harmonic function.

$$\text{Let } v = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial x} = - \frac{[(x^2 + y^2) \cdot 0 - y(2x)]}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2 + y^2)^2 (2y) - 2xy \cdot 2(x^2 + y^2)(2x)}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)^2 2y - 8x^2 y (x^2 + y^2)}{(x^2 + y^2)^4}$$

$$= \frac{2y(x^2 + y^2) - 8x^2 y}{(x^2 + y^2)^3}$$

$$= \frac{2y^3 - 6x^2 y}{(x^2 + y^2)^3}$$

$$\frac{\partial v}{\partial y} = - \frac{[(x^2 + y^2) - y \cdot 2y]}{(x^2 + y^2)^2} = \frac{-(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2 + y^2)^2 2y - (y^2 - x^2) 2(x^2 + y^2) 2y}{(x^2 + y^2)^4}$$





## DEPARTMENT OF MATHEMATICS

$$u_{xx} + u_{yy} = 0$$

$\therefore u$  satisfies Laplace equation.

$\Rightarrow u$  is harmonic

$$\text{Now } v = \int \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

$$= \int (6xy + 6y) dx + (3x^2 - 3y^2 + 6x) dy + c$$

$$= \frac{6x^2y}{2} + 6xy + 3x^2y - \frac{3y^3}{3} + 6xy + c$$

$$= \frac{1}{2} [6x^2y + 12xy + 6x^2y - 2y^3 + 12xy + 2c]$$

$$v = 6x^2y + 12xy - y^3 + c$$

- ③ ST  $u = \cos x \cosh y$  is harmonic & hence find its harmonic conjugate.

Soln:

$$u = \cos x \cosh y$$

$$u_x = -\sin x \cosh y \quad ; \quad u_y = \cos x \sinh y$$

$$u_{xx} = -\cos x \cosh y \quad , \quad u_{yy} = \cos x \cosh y$$

$$u_{xx} + u_{yy} = 0$$

$\Rightarrow u$  satisfies Laplace eqn

$\Rightarrow u$  is harmonic.

$$\text{Now } v = \int \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

$$= \int (-\cos x \sinh y dx) + (-\sin x \cosh y) dy + c$$

$$= -\sin x \sinh y - \sin x \sinh y + c$$

$$v = -2 \sin x \sinh y + c$$