Construction of Analytic function:

Milne-Thomson method:

Let $f(z) = u+iv$ is to be constructed

(i) Suppose the real part u is given, then

$$f(z) = \int [\varphi_1(z,0) - i\varphi_2(z,0)]\,dz$$

where $\varphi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$, $\varphi_2(z,0) = \frac{\partial u}{\partial y}(z,0)$.

(ii) Suppose imaginary part v is given, then

$$f(z) = \int [\varphi_1(z,0) + i\varphi_2(z,0)]\,dz$$

where $\varphi_1(z,0) = \frac{\partial v}{\partial y}(z,0)$, $\varphi_2(z,0) = \frac{\partial v}{\partial x}(z,0)$

(1) Show that the fn $u = x^3 + x^2 - 3xy^2 + 2xy - y^2$ is harmonic and find the corresponding analytic fn $f(x) = u + iv$.

Soln:

$$u = x^3 + x^2 - 3xy^2 + 2xy - y^2$$

$$u_x = 3x^2 + 2x - 3y^2 + 2y \quad ; \quad u_y = -6xy + 2x - 2y$$

$$u_{xx} = 6x + 2 \quad ; \quad u_{yy} = -6x - 2$$

$$u_{xx} + u_{yy} = 0$$

$\Rightarrow$ u is harmonic. $\quad f(z) = \int \left[\frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)\right] dz$

$$f(z) = \int [\varphi_1(z,0) - i\,\varphi_2(z,0)]\,dz$$

where $\varphi_1(z,0) = \frac{\partial u}{\partial x}(z,0) = 3z^2 + 2z$

$$\varphi_2(z,0) = \frac{\partial u}{\partial y}(z,0) = 2z$$

$$f(z) = \int [(3z^2 + 2z) - i \cdot 2z]\,dz$$

$$= \frac{3z^3}{3} + \frac{2z^2}{2} - i \cdot \frac{2z^2}{2} + c$$

$$f(z) = z^3 + z^2(1 - i) + c.$$

(2) Find an analytic fn whose imaginary part is $v = e^{2x}(y\cos 2y + x\sin 2y)$.

Soln:

$$v = e^{2x}(y\cos 2y + x\sin 2y)$$

$$v_x = 2e^{2x}(y\cos 2y + x\sin 2y) + e^{2x}\sin 2y$$

$$v_y = e^{2x}(\cos 2y + y(-2\sin 2y) + 2x\cos 2y)$$

$$f(z) = \int [\varphi_1(z,0) + i\,\varphi_2(z,0)]\,dz$$

$$= \int \left[\frac{\partial v}{\partial y}(z,0) + i\,\frac{\partial v}{\partial x}(z,0)\right] dz$$

(3) Prove that $u = x^2 - y^2$, $v = \dfrac{-y}{x^2+y^2}$ are harmonic but $u+iv$ is not a regular function.

Soln:

Let $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

$\Rightarrow u$ is a harmonic function.

Let $v = \dfrac{-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = -\frac{[(x^2+y^2)0 - y(2x)]}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2\, 2y - 8x^2y\,(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = -\frac{[(x^2+y^2) - y \cdot 2y]}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2\, 2y - (y^2-x^2)\, 2(x^2+y^2)\, 2y}{(x^2+y^2)^4}$$

$\varphi_1(z,0) = \frac{\partial v}{\partial y}(z,0) = e^{2z}(1+0+2z)$

$= e^{2z} + 2ze^{2z}$

$\varphi_2(z,0) = \frac{\partial v}{\partial x}(z,0) = 2e^{2z}(0) + e^{2z}(0) = 0.$

$f(z) = \int [e^{2z} + 2ze^{2z}]\,dz + c$

$= \int e^{2z}\,dz + 2\int ze^{2z}\,dz + c$

$= \frac{e^{2z}}{2} + 2\left(\frac{ze^{2z}}{2} - \frac{e^{2z}}{4}\right) + c$

$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$

$= ze^{2z} + c.$

(3) Given that $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$, find the analytic fn whose real part is u

Soln:

$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$u_x = \frac{(\cosh 2y - \cos 2x)\cos 2x \cdot 2 - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

$u_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$

$f(z) = \int [\varphi_1(z,0) - \varphi_2(z,0)]\,dz$

$\varphi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$

$= \frac{(\cos 0 - \cos 2z)\,2\cos 2z - \sin 2z\,(2\sin 2z)}{(\cos 0 - \cos 2z)^2}$

$$= \frac{2\cos 2z - (2\cos^2 2z - 2\sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\varphi_2(z,0) = \frac{(\cos 0 - \cos 2z)(0) - \sin 2z(2\sin 0)}{(\cos 0 - \cos 2z)^2}$$

$$= 0.$$

$$f(z) = \int -\operatorname{cosec}^2 z \, dz = \cot z + c$$

$$f(z) = \cot z + c$$