



Cross Spectral Density (or) Cross Power Spectral Density

Consider two jointly WSS random processes $x(t)$ and $y(t)$.

Let $R_{xy}(\tau)$ and $R_{yx}(\tau)$ be their cross correlation functions.

Then the Cross Spectral densities are,

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} d\tau$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-i\omega\tau} d\tau$$



Properties:

1] $S_{xy}(\omega) = S_{yx}(-\omega)$

Proof:

$$\text{Now, } S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{yx}(-\tau) e^{-i\omega\tau} d\tau$$

$$\therefore R_{xy}(\tau) = R_{yx}(-\tau)$$

$$= \int_{\infty}^{-\infty} R_{yx}(\tau_1) e^{i\omega\tau_1} (-d\tau_1) \quad \left| \begin{array}{l} \text{Put } \tau_1 = -\tau \\ \text{when } \tau = \infty \Rightarrow \tau_1 = -\infty \\ \tau = -\infty \Rightarrow \tau_1 = \infty \end{array} \right.$$

$$= \int_{-\infty}^{\infty} R_{yx}(\tau_1) e^{i\omega\tau_1} d\tau_1$$

$$= S_{yx}(-\omega)$$

$$\therefore S_{xy}(\omega) = S_{yx}(-\omega)$$

2] Real part of $S_{xy}(\omega)$ is an even function of ω . i.e., $S_{xy}(\omega) = S_{xy}(-\omega)$

Proof:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) (\cos \omega\tau - i \sin \omega\tau) d\tau$$

$$\text{Real part of } S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) \cos \omega\tau d\tau$$



$$\operatorname{Re} [S_{xy}(-\omega)] = \int_{-\infty}^{\infty} R_{xy}(\tau) \cos(-\omega)\tau \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \cos \omega\tau \, d\tau$$

$$= \operatorname{Re} [\phi_{xy}(\omega)] \quad \because \cos(-\theta) = \cos \theta$$

$\therefore \operatorname{Re} [S_{xy}(\omega)]$ is an even function of ω .

3]. Imaginary part of $S_{xy}(\omega)$ is an odd function of ω .

Proof :

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) (\cos \omega\tau - i \sin \omega\tau) \, d\tau$$

$$\operatorname{Im} [\phi_{xy}(\omega)] = - \int_{-\infty}^{\infty} R_{xy}(\tau) \sin \omega\tau \, d\tau \rightarrow (1)$$

$$\operatorname{Im} [\phi_{xy}(-\omega)] = - \int_{-\infty}^{\infty} R_{xy}(\tau) \sin(-\omega\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \sin(\omega\tau) \, d\tau$$

$$[\because \sin(-\theta) = -\sin \theta]$$

$$= - \operatorname{Im} [S_{xy}(\omega)] \quad [\text{By (1)}]$$

$\therefore \operatorname{Im} [S_{xy}(\omega)]$ is an odd function of ω .



11. The cross power spectrum of a real narrow band process $x(t)$ and $y(t)$ is given by,

$$S_{xy}(\omega) = \begin{cases} a + \frac{ib\omega}{\alpha}, & -\alpha < \omega < \alpha, \alpha > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the cross correlation function.

Soln.

$$\begin{aligned} R_{xy}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{i\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \left(a + \frac{ib\omega}{\alpha}\right) e^{i\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} a e^{i\omega\tau} d\omega + \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \frac{ib\omega}{\alpha} e^{i\omega\tau} d\omega \end{aligned}$$

$$= \frac{a}{2\pi} \left[\frac{e^{i\omega\tau}}{i\tau} \right]_{-\alpha}^{\alpha} + \frac{ib}{2\pi\alpha} \left[\omega \left(\frac{e^{i\omega\tau}}{i\tau} \right) - \left(\frac{e^{i\omega\tau}}{(i\tau)^2} \right) \right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{2\pi i\tau} [e^{i\alpha\tau} - e^{-i\alpha\tau}] + \frac{ib}{2\pi\alpha} \left[\left(\frac{\alpha}{i\tau} e^{i\alpha\tau} + \frac{e^{i\alpha\tau}}{\tau^2} \right) - \left(-\frac{\alpha}{i\tau} e^{-i\alpha\tau} + \frac{e^{-i\alpha\tau}}{\tau^2} \right) \right]$$

$$= \frac{a}{2\pi i\tau} [2i \sin \alpha\tau] + \frac{ib}{2\pi\alpha} \left[\frac{\alpha}{i\tau} (e^{i\alpha\tau} - e^{-i\alpha\tau}) + \frac{1}{\tau^2} (e^{i\alpha\tau} - e^{-i\alpha\tau}) \right]$$

$$= \frac{a}{\pi\tau} \sin \alpha\tau + \frac{ib}{2\pi\alpha} \left[\frac{\alpha}{i\tau} 2 \cos \alpha\tau + \frac{1}{\tau^2} 2i \sin \alpha\tau \right]$$

$$= \frac{a}{\pi\tau} \sin \alpha\tau + \frac{b}{\pi\tau} \cos \alpha\tau - \frac{b}{\pi\alpha\tau^2} \sin \alpha\tau$$



Q). If the cross correlation of two processes $\{x(t)\}$ and $\{y(t)\}$ is $R_{xy}(t, t+\tau) =$

$$\frac{AB}{2} [\sin(\omega_0 \tau) + \cos(\omega_0 (2t + \tau))]$$

where A, B are ω_0 and constants. find the cross power spectrum.

Soln.

The time average is given by,

$$R_{xy}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{xy}(t, t+\tau) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{AB}{2} [\sin \omega_0 \tau + \cos \omega_0 (2t + \tau)] dt$$

$$= \frac{AB}{2} \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \sin(\omega_0 \tau) dt + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos \omega_0 (2t + \tau) dt \right]$$

$$= \frac{AB}{2} \left[\sin \omega_0 \tau + \lim_{T \rightarrow \infty} \frac{1}{4T} \int_{-T}^T \sin(\omega_0 (2t + \tau)) dt \right]$$

$$= \frac{AB}{2} \sin \omega_0 \tau + 0$$

The cross spectrum is

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} \frac{AB}{2} \sin \omega_0 \tau e^{-i\omega \tau} d\tau$$

$$= \frac{AB}{2} \int_{-\infty}^{\infty} \sin \omega_0 \tau (\cos \omega \tau - i \sin \omega \tau) d\tau$$



$$\begin{aligned}
 &= \frac{AB}{2} \left[\int_{-\infty}^{\infty} \sin \omega_0 \tau \cos \omega \tau \, d\tau - i \int_{-\infty}^{\infty} \sin \omega_0 \tau \sin \omega \tau \, d\tau \right] \\
 &= \frac{AB}{2} \left[\frac{1}{2} \int_{-\infty}^{\infty} [\sin(\omega_0 + \omega) \tau + \sin(\omega_0 - \omega) \tau] \, d\tau - \right. \\
 &\quad \left. \frac{i}{2} \int_{-\infty}^{\infty} [\cos(\omega - \omega_0) \tau - \cos(\omega + \omega_0) \tau] \, d\tau \right] \\
 &= \frac{AB}{4} \left[\int_{-\infty}^{\infty} \sin(\omega_0 + \omega) \tau + \sin(\omega_0 - \omega) \tau - i \cos(\omega - \omega_0) \tau \right. \\
 &\quad \left. + i \cos(\omega + \omega_0) \tau \, d\tau \right] \\
 &= \frac{AB}{4} \int_{-\infty}^{\infty} \{ i [\cos(\omega - \omega_0) \tau - i \sin(\omega - \omega_0) \tau] \\
 &\quad - i [\cos(\omega + \omega_0) \tau - i \sin(\omega + \omega_0) \tau] \} \, d\tau \\
 &= -\frac{iAB}{4} \int_{-\infty}^{\infty} [e^{-i(\omega - \omega_0) \tau} - e^{-i(\omega + \omega_0) \tau}] \, d\tau \\
 &= -\frac{iAB}{4} \left[\int_{-\infty}^{\infty} e^{-i(\omega - \omega_0) \tau} \, d\tau - \int_{-\infty}^{\infty} e^{-i(\omega + \omega_0) \tau} \, d\tau \right] \\
 &= -\frac{iAB}{4} [2\pi \delta(\omega - \omega_0) - 2\pi \delta(\omega + \omega_0)] \\
 &= -\frac{i\pi AB}{2} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]
 \end{aligned}$$



3. Given the Power Spectral density $S_{xx}(\omega) = \frac{\omega^2 + 9}{\omega^2 + 5\omega + 4}$.
Find the mean square value of the process.

Soln.

$$\text{Given } S_{xx}(\omega) = \frac{\omega^2 + 9}{\omega^2 + 5\omega + 4}$$

$$\text{put } u = \omega^2$$

$$S_{xx}(\omega) = \frac{u+9}{u^2+5u+4} = \frac{u+9}{(u+1)(u+4)} = \frac{A}{u+1} + \frac{B}{u+4}$$

$$u+9 = A(u+4) + B(u+1)$$

$$\text{put } u = -1, \quad -1+9 = A(-1+4) + 0 \Rightarrow A = 8/3$$

$$u = -4, \quad -4+9 = 0 + B(-4+1) \Rightarrow B = -5/3$$

$$\therefore S_{xx}(\omega) = \frac{8}{3} \frac{1}{u+1} - \frac{5}{3} \frac{1}{u+4} = \frac{8}{3} \frac{1}{\omega^2+1} - \frac{5}{3} \frac{1}{\omega^2+4}$$

To find:

mean square value of the process i.e., $R_{xx}(0)$

WKT

$$R_{xx}(\tau) = F^{-1}(S_{xx}(\omega))$$

$$= F^{-1} \left[\frac{8}{3} \frac{1}{\omega^2+1} - \frac{5}{3} \frac{1}{\omega^2+4} \right]$$

$$= \frac{8}{3} \cdot \frac{1}{2} F^{-1} \left(\frac{2}{\omega^2+1^2} \right) - \frac{5}{3} \cdot \frac{1}{4} F^{-1} \left(\frac{2 \cdot 2}{\omega^2+2^2} \right)$$

$$= \frac{8}{6} e^{-|\tau|} - \frac{5}{12} e^{-2|\tau|}$$

$$\text{Now, } R_{xx}(0) = \frac{8}{6} - \frac{5}{12}$$

$$= \frac{11}{12}$$