Unit- IV

Complex Integration

Cauchy's integral theorem (or) Cauchy's fundamental theorem:

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all Points inside and on a simple closed curve c, then

$$\boxed{\int_C f(z)\,dz = 0}$$

Cauchy's integral formula:

If $f(z)$ is analytic inside and on a simple closed curve 'c' and 'a' be any Point inside c then

$$\boxed{\int_C \frac{f(z)\,dz}{z-a} = 2\pi i\, f(a)}$$

Note: If the point lies outside c then $\int_C \frac{f(z)}{z-a}\,dz = 0$

Cauchy's integral formula for derivatives:

If $f(z)$ is analytic inside and on a simple closed curve 'c' and 'a' be any Point inside c then,

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i}{1!} f'(a)$$

$$\int_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

In general,
$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$$

Problems :

1) Evaluate $\int_C \frac{\cos \pi z}{z-1} dz$ if c is $|z| = 2$.

Sol: Formula :

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$z-a=0$

$z-1=0$

$z=1$

Given : $|z|=2$

$1<2$

It lies inside c.

(Here $a=1$

$f(z) = \cos \pi z$)

$$\int_C \frac{\cos \pi z}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i \cdot \cos \pi$$

$$= 2\pi i (-1)$$

$$= -2\pi i$$

$$\boxed{\int_C \frac{\cos \pi z}{z-1} dz = -2\pi i}$$

2) Evaluate $\int_C \frac{z dz}{z-2}$ where c is the circle $|z|=1$.

Sol: $f(z) = \frac{z}{z-2}$

$z-2=0$

$z=2$ $\quad$ ($\because |z|=1$, $2>1$)

Here $z=2$ lies outside c.

$$\int_C f(z) dz = 0$$

$$\boxed{\int_C \frac{z\, dz}{z-2} = 0}$$

3) Evaluate $\int_C \frac{z}{(z-1)^3} dz$ where c is $|z|=2$.

Sol: $z-1=0$

$z=1$

Given : $|z|=2 \Rightarrow 1<2$

$\therefore$ It lies inside c.

Formula:- $\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$

$\int_c \frac{f(z)}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$

$= \frac{\cancel{2}\pi i}{\cancel{2}} f''(1)$

$= \pi i f''(1)$

$= \pi i (0)$

$= 0$

$f(z) = z$

$f'(z) = 1$

$f''(z) = 0$

$\therefore f''(1) = 0$

$$\boxed{\therefore \int_c \frac{f(z)}{(z-1)^3} dz = 0}$$

4) Using Cauchy's integral formula evaluate $\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ where c is $|z| = \frac{3}{2}$.

Sol:- Using Partial fractions,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \longrightarrow (1)$$

$1 = A(z-2) + B(z-1)$

Put $z=1 \Rightarrow 1 = A(1-2) + 0$

$1 = A(-1)$

$\boxed{A = -1}$

Put $z=2 \Rightarrow 1 = A(0) + B(2-1)$

$1 = 0 + B$

$\boxed{B = 1}$

$(1) \Rightarrow \dfrac{1}{(z-1)(z-2)} = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$

Multiply by $\cos \pi z^2$ on both sides, we get

$$\frac{\cos \pi z^2}{(z-1)(z-2)} = \frac{-\cos \pi z^2}{z-1} + \frac{\cos \pi z^2}{z-2}$$

$$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{-\cos \pi z^2}{z-1} dz + \int_C \frac{\cos \pi z^2}{z-2} dz \rightarrow (2)$$

$z = 1 \; ; \; z = 2$

$1 < \frac{3}{2} \; ; \; 2 > \frac{3}{2}$

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i f(1) + 0$$

$$= -2\pi i(-1)$$

$$= 2\pi i$$

$f(z) = \cos \pi z^2$

$f(1) = \cos \pi$

$= -1$

$$\boxed{\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i}$$

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5) Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is $|z| = 3$.

Sol: Using Partial fractions,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow (1)$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1 \Rightarrow 1 = A(1-2) + B(1-1)$

$$\boxed{-1 = A}$$

Put $z=2 \Rightarrow 1 = A(2-2)+B(2-1)$

$1 = 0+B$

$\boxed{1=B}$

$$(1) \Rightarrow \frac{1}{(z-1)(z-2)} = \frac{A}{z-1}+\frac{B}{z-2} = \frac{-1}{z-1}+\frac{1}{z-2}$$

$$\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz = \int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\left(\frac{-1}{z-1}+\frac{1}{z-2}\right)dz$$

$$= -\int_c \frac{\sin\pi z^2+\cos\pi z^2}{z-1}\,dz + \int_c \frac{\sin\pi z^2+\cos\pi z^2}{z-2}\,dz$$

$$= -[(2\pi i(0-1))] + 2\pi i(0+1)$$

$z=1$
$z=2$

$$= -(-2\pi i)+2\pi i$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

$$\boxed{\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz = 4\pi i}$$