Taylor's Series:

If $f(z)$ is analytic inside a circle c with centre at $z=a$ then $f(z)$ canbe expressed as,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a)$$

which is convergent at every point inside c. This is called Taylor's Series of $f(z)$ about $z=a$.

Note:

The Taylor's Series of $f(z)$ about the point $z=0$ is given by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \cdots + \frac{z^n}{n!} f^n(0) + \cdots$$

Problems:

1) Expand $f(z) = \log(1+z)$ as Taylor's Series about $z=0$ if $|z|<1$.

Sol: $f(z) = \log(1+z)$ $\quad$ $f(0) = \log 1 = 0$

$f'(z) = \frac{1}{1+z}$ $\quad$ $f'(0) = \frac{1}{1+0} = 1$

$f''(z) = \frac{-1}{(1+z)^2}$ $\quad$ $f''(0) = \frac{-1}{1} = -1$

$f'''(z) = \frac{2}{(1+z)^3}$ $\quad$ $f'''(0) = \frac{2}{1} = 2$

Taylor's series about $z=0$ is given by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots$$

$$= 0 + \frac{z}{1!}(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(2) + \cdots\cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\cdots$$

2) Expand $f(z) = \cos z$ about $z = \frac{\pi}{3}$ in Taylor's series.

Sol:. $f(z) = \cos z$ $\qquad f(\pi/3) = \cos\frac{\pi}{3} = \frac{1}{2}$

$f'(z) = -\sin z$ $\qquad f'(\pi/3) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$f''(z) = -\cos z$ $\qquad f''(\pi/3) = -\cos\frac{\pi}{3} = \frac{-1}{2}$

$f'''(z) = \sin z$ $\qquad f'''(\pi/3) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Taylor's Series is

$$f(z) = f(\pi/3) + \frac{(z-\pi/3)}{1!}f'\left(\frac{\pi}{3}\right) + \frac{(z-\pi/3)}{2!}f''\left(\frac{\pi}{3}\right) + \cdots\cdots$$

$$= \frac{1}{2} + \frac{(z-\pi/3)}{1!}\left(-\frac{\sqrt{3}}{2}\right) + \frac{(z-\pi/3)^2}{2!}\left(\frac{-1}{2}\right) + \frac{(z-\pi/3)^3}{3!}\left(\frac{\sqrt{3}}{2}\right) + \cdots$$

$$= \frac{1}{2} - \frac{(z-\pi/3)}{1!}\left(\frac{\sqrt{3}}{2}\right) - \frac{(z-\pi/3)^2}{2!}\left(\frac{1}{2}\right) + \frac{(z-\pi/3)}{3!}\left(\frac{\sqrt{3}}{2}\right) + \cdots$$

3) Expand $f(z) = \sin z$ in a Taylor's Series about $z = \frac{\pi}{4}$.

Sol:. $f(z) = \sin z$ $\qquad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z$ $\qquad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z$ $\qquad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

$f'''(z) = -\cos z$ $\qquad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

Taylor Series formula for $z = a$

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) - \cdots$$

At $z = \frac{\pi}{4}$

$$f(z) = f\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)}{1!} f'\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(z-\frac{\pi}{4}\right)^2}{2}\left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{6}\left(\frac{-1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3 + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3 + \cdots$$

Zeros of an analytic function:

If a function $f(z)$, analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called a zero of $f(z)$.

Simple zero:

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $z = z_0$ is called a simple zero of $f(z)$ or a zero of the first order.

Zero of order n:

If $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$ then z_0 is called zero of order n.

(or)

An analytic function $f(z)$ is said to have a zero of order n if $f(z)$ can be expressed as $f(z) = (z-z_0)^m \phi(z)$, where $\phi(z)$ is analytic and $\phi(z_0) \neq 0$.