4) Find the zeros of $\frac{\sin z - z}{z^3}$

Sol: Let $f(z) = \frac{\sin z - z}{z^3}$

Zeros of $f(z)$ are given by $f(z) = 0$

$$f(z) = \frac{\sin z - z}{z^3} = \frac{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots\right] - z}{z^3}$$

$$= \frac{-\frac{z^3}{3!} + \frac{z^5}{5!} - \dots}{z^3}$$

$$= -\frac{1}{3!} + \frac{z^2}{5!} - \dots$$

$$\lim_{z \to 0} \frac{\sin z - z}{z^3} \neq 0$$

$f(z)$ has no zeros.

5) Find the zeros of $\frac{1 - e^{2z}}{z^4}$

Sol: $f(z) = \frac{1 - e^{2z}}{z^4}$

zeros of $f(z)$ are given by $f(z) = 0$

$$\frac{1 - e^{2z}}{z^4} = 0$$

$$1 - e^{2z} = 0$$

$$e^{2z} = e^{2in\pi}$$

$$2z = 2in\pi$$

$$z = in\pi \Rightarrow 0, \pm\pi, \pm 2\pi, \dots$$

Singularities:

A Point $z = z_0$ at which a function $f(z)$ fails to be analytic is called a Singular Point or singularity of $f(z)$.

Eg: $f(z) = \frac{1}{z-3}$

Here $z = 3$ is a singular Point of $f(z)$.

Types of Singularities:

i) Isolated Singularity:

A Point $z = z_0$ is said to be isolated singularity of $f(z)$ if

i) $f(z)$ is not analytic at $z = z_0$

ii) There exists a neighbourhood of $z = z_0$ containing no other singularity.

Eg: $f(z) = \frac{1}{z}$

It is analytic everywhere except at $z = 0$.

$z = 0$ is an isolated singularity.

Note: If $z = z_0$ is an isolated singular Point of a function $f(z)$, then the singularity is called

(i) a removable singularity (or) (ii) a Pole (or)

(iii) an essential singularity

According to Laurent's Series about $z = z_0$ of $f(z)$ Valid in a deleted neighbourhood of $z = z_0$ has

i) no negative powers or

ii) a finite number of negative powers or

iii) an infinite number of negative powers.

Note:2 Let $z = z_0$ be an isolated singular point of the function $f(z)$. Then $f(z)$ is analytic inside and on a circle c except at $z = z_0$, the centre of c and therefore $f(z)$ can be expanded by a Laurent's Series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} b_n (z-z_0)^{-n}$$

Removable Singularity :-

A singular point $z = z_0$ is called a removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists finitely.

$$f(z) = \frac{\sin z}{z}$$

$$\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \cdots\right]$$

$$= 1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} - \cdots$$

there is no negative powers of z.

$z = 0$ is a removable singularity of $f(z)$.

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\sin z}{z} = 1 \quad \text{(finite)}$$

$z = 0$ is a removable singularity.

Problems:

1) what is the nature of the singularity $z=0$ of the function $f(z) = \frac{\sin z - z}{z^3}$

Sol:. $f(z) = \frac{\sin z - z}{z^3}$

A function $f(z)$ is not defined at $z=0$.

By L'Hospital's rule, $\lim\limits_{z\to 0} \frac{\sin z - z}{z^3} = \lim\limits_{z\to 0} \frac{\cos z - 1}{3z^2} = \lim\limits_{z\to 0} \frac{-\sin z}{6z}$

$$= \lim_{z\to 0} \frac{-\cos z}{6}$$

$$= -\frac{1}{6}$$

Since the limit exists and finite, the singularity at $z=0$ is a removable singularity.

2) Find the nature of the singularity at $z=0$ of $f(z) = \frac{\sin z}{z}$.

Sol:. $f(z) = \frac{\sin z}{z}$

A function $f(z)$ is not defined at $z=0$

By L'Hospital's rule.

$$\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z\to 0} \frac{\cos z}{1} = 1$$

since the limit exists and is finite, the singularity at $z=0$ is a removable singularity.

Poles:.

An analytic function $f(z)$ with a singularity at $z=a$ if $\lim_{z \to a} f(z) = \infty$ then $z=a$ is a Pole of $f(z)$.

Simple Pole:. A Pole of order one is called a simple Pole.

Eg:. If $f(z) = \dfrac{1}{(z-4)^2(z-3)^3(z-1)}$

Here $z=1$ is a simple pole

$z=3$ is a pole of order 3

$z=4$ is a pole of order 2

Essential Singularity:.

If the Principal Part Contains an infinite number of non zero terms, then $z=z_0$ is known as a essential singularity.

Eg: $f(z) = e^{\frac{1}{z}} = 1 + \dfrac{\frac{1}{z}}{1!} + \dfrac{\left(\frac{1}{z}\right)^2}{2!} + \_\_\_\_$ has $z=0$ as an essential singularity.

Since $f(z)$ is an infinite series of -ve powers of z.

$f(z) = e^{\frac{1}{z-4}}$ has $z=4$ as an essential singularity.

Entire function / Integral function:

A function $f(z)$ which is analytic everywhere in the finite plane (except at infinity) is called an entire function or an integral function.

Eg:. e^z, $\sin z$, $\cos z$.