Meromorphic function:

A function f(z) which is analytic everywhere in the finite plane except at finite number of poles is called a meromorphic function.

Eg: $f(z) = \frac{z+3}{(z-1)(z-2)^2}$.

Laurent's Series:

Let c_1 and c_2 be two concentric circles then $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

Let f(z) be analytic inside and on the annular region R between c_1 and c_2. Then for any $z \in R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n f(z) dz$$

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{f(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

Problems:

1) Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in a Laurent's series if

(i) $|z| < 2$ (ii) $|z| > 3$ and (iii) $2 < |z| < 3$.

Sol: Using Partial fractions,

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3} \rightarrow (1)$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A(z+2)(z+3)+B(z+3)+C(z+2)}{(z+2)(z+3)}$$

$$z^2-1 = A(z+2)(z+3)+B(z+3)+C(z+2)$$

$$z=-2 \Rightarrow (-2)^2-1 = A(0)+B(-2+3)+0$$

$$4-1 = B$$

$$\boxed{B=3}$$

$$z=-3 \Rightarrow (-3)^2-1 = 0+0+C(-3+2)$$

$$9-1 = -C$$

$$8 = -C$$

$$\boxed{C=-8}$$

$$z=0 \Rightarrow 0-1 = A(2)(3)+B(3)+C(2)$$

$$-1 = 6A+3B+2C$$

$$-1 = 6A+9-16$$

$$-1 = 6A-7$$

$$6A = 7-1$$

$$6A = 6$$

$$\boxed{A=1}$$

$$(1) \Rightarrow f(z) = 1+\frac{3}{z+2}-\frac{8}{z+3}$$

$$(i)\ |z|<2 \Rightarrow \frac{|z|^2}{3}<1$$

i) $|z| < 2 \Rightarrow \frac{|z|}{2} < 1$

$$(2) \Rightarrow f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right]$$

$$- \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right]$$

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$$

ii) $|z| > 3$

$3 < |z|$

$\frac{3}{|z|} < 1$

$$(2) \Rightarrow f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \cdots\right] - \frac{8}{z}\left(1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \cdots\right]$$

iii) $2 < |z| < 3$

$2 < |z|$ and $|z| < 3$

$\frac{2}{|z|} < 1$ and $\frac{|z|}{3} < 1$

(2) $\Rightarrow 1 + \dfrac{3}{z\left(1+\frac{2}{z}\right)} - \dfrac{8}{3\left(1+\frac{z}{3}\right)}$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 \cdots\right)$$

2) Expand $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$ in Laurent's Series if

i) $|z| < 2$ ii) $|z| > 3$ iii) $2 < |z| < 3$ iv) $1 < |z+1| < 3$.

Sol: Given $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

when $z=0$, $A = 1$

$z=-1$, $C = -3$

$z = 2$, $B = 2$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

i) $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{-2\left(1-\frac{z}{2}\right)} - \frac{3}{z+1}$$

$$= \frac{1}{z} - \left(1-\frac{z}{2}\right)^{-1} - 3(z+1)^{-1}$$

$$= \frac{1}{z} - \left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^3+\dots\right] - 3\left[1-z+z^2-z^3+\dots\right]$$

ii) $|z|>3 \Rightarrow \frac{3}{|z|}<1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z\left(1-\frac{2}{z}\right)} - \frac{3}{z\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\dots\right] - \frac{3}{z}\left(1-\frac{1}{z}+\dots + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \dots\right]$$

iii) $2<|z|<3$

$|z|>2, \ |z|<3$

$\frac{2}{|z|}<1, \ \frac{|z|}{3}<1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z\left(1-\frac{2}{z}\right)} - \frac{3}{z\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right) - \frac{3}{z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2 \cdots\right)$$

iv) $1 < |z+1| < 3$

Let $u = z+1$

$z = u-1$

$1 < |u| < 3$

$\frac{1}{|u|} < 1, \quad \left|\frac{u}{3}\right| < 1$

$$f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{-3\left(1-\frac{u}{3}\right)} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\cdots\right) - \frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots\right) - \frac{3}{u}$$

$$f(z) = \frac{1}{z+1}\left(1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\cdots\cdots\right) - \frac{2}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\cdots\right) - \frac{3}{z+1}$$