Residues:.

If $z = z_0$ is an isolated singular point of $f(z)$, we can find the Laurent Series of $f(z)$ about $z = z_0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

The coefficient of $\frac{1}{z-z_0}$ in the above expansion is called the residue of $f(z)$ at $z = z_0$.

ie,) b_1 is the residue of $f(z)$ at $z = z_0$.

From a definition of b_n,

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$

Residue of $f(z)$ at $z = z_0$ may be denoted by Res $[f(z), z_0]$.

Evaluation of Residues:.

i) Residue at a Pole of order m:.

If $z = z_0$ is a Pole of order m, a simple formula to determine the residue is given by

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

ii) Residue at a Simple Pole:

If $z = z_0$ is a Simple Pole

If $m=1$, the residues to

$$b_1 = \lim_{z \to z_0} (z-z_0) f(z)$$

iv If $z=z_0$ is a simple Pole of $f(z)$, then

$$\text{Res}\left[f(z), z_0\right] = \lim_{z \to z_0} (z-z_0) f(z).$$

iii) If $z=z_0$ is a simple pole of $f(z)$ and if

$$f(z) = \frac{\phi(z)}{\psi(z)} \text{ then } \text{Res}\left[f(z), z_0\right] = \frac{\phi(z_0)}{\psi'(z_0)}$$

Problems:

1) Find the residue of $f(z) = \dfrac{z}{(z-1)^2}$ at its Pole.

Sol: Given $f(z) = \dfrac{z}{(z-1)^2}$

Here $z=1$ is a Pole of order 2.

$$\text{Re}(z=z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

$$\text{Re}(z=1) = \lim_{z \to 1} \frac{1}{1!} \frac{d}{dz}\left(\cancel{(z-1)^2} \frac{z}{\cancel{(z-1)^2}}\right)$$

$$= \lim_{z \to 1} \frac{d}{dz}(z)$$

$$= \lim_{z \to 1} (1)$$

$$= 1$$

2) Find the residue of $\cot z$ at $z=0$.

Sol: $f(z) = \cot z = \frac{\cos z}{\sin z} = \frac{\phi(z)}{\psi(z)}$

Here $z=0$ is a simple pole.

$\phi(z) = \cos z$ $\quad\quad$ $\psi(z) = \sin z$

$\phi(0) = \cos 0$ $\quad\quad$ $\psi'(z) = \cos z$

$= 1$ $\quad\quad$ $\psi'(0) = 1$

$$Re[f(z), 0] = \frac{\phi(0)}{\psi'(0)} = \frac{1}{1} = 1.$$

3) Find the residue of $f(z) = \frac{1}{z^2 e^z}$

Sol: $f(z) = \frac{e^{-z}}{z^2} = \frac{1 - z + \frac{z^2}{\lfloor 2} - \frac{z^3}{\lfloor 3} + \cdots}{z^2}$

$$= \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots$$

$Re(f(z), 0)$ = coefficient of $\frac{1}{z}$ in Laurent expansion

$Re(f(z), 0) = -1$

4) Find the residues at $z=0$ of the functions

i) $f(z) = e^{1/z}$ $\quad$ ii) $f(z) = \frac{\sin z}{z^4}$ $\quad$ iii) $f(z) = z \cos \frac{1}{z}$.

Sol: The residues are the coefficients of $\frac{1}{z}$ in the Laurent's expansion of $f(z)$ about $z=0$.

(i.e.) i) $e^{\frac{1}{z}} = 1 + \frac{\frac{1}{z}}{\lfloor 1} + \frac{(\frac{1}{z})^2}{\lfloor 2} + \cdots$

$$= 1 + \frac{1}{\lfloor 1}\left(\frac{1}{z}\right) + \frac{1}{\lfloor 2}\frac{1}{z^2} + \frac{1}{\lfloor 3}\frac{1}{z^3} + \cdots$$

Res $(f(z),0)$ = Coefficient of $\frac{1}{z}$ in Laurent's expansion of $f(z)$ about $z=0$.

$$\text{Res}[f(z),0] = \frac{1}{L1} = 1.$$

ii) $$\frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} \cdots$$

Re $(f(z),0)$ = Coeff. of $\frac{1}{z}$ in Laurent's expansion

$$\text{Res}(f(z),0) = -\frac{1}{3!} = \frac{-1}{6}.$$

iii) $$z\cos\frac{1}{z} = z\left(1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \cdots\right]$$

$$= z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \cdots$$

Res$[f(z),0]$ = Coef. of $\frac{1}{z}$ in Laurent's expansion

$$= \frac{-1}{2!}$$

$$= \frac{1}{2}.$$