Cauchy-Residue theorem :-

If $f(z)$ is analytic at all points inside and on a simple closed curve C except at a finite number of points $z_1, z_2, \ldots z_n$ inside C, then $\int_C f(z)\,dz = 2\pi i$ (Sum of residues of $f(z)$ at $z_1, z_2, \ldots z_n$)

Proof :-

Given that $f(z)$ is not analytic only at $z_1, z_2, \ldots z_n$.

Draw the non-intersecting small circles $C_1, C_2, \ldots C_n$ with centre at $z_1, z_2, \ldots z_n$ and radii $\rho_1, \rho_2, \ldots \rho_n$ lying wholly inside C.

Then $f(z)$ is analytic in the region between C and $C_1, C_2, \ldots C_n$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz \quad \rightarrow (1)$$

Now, $z_1, z_2, \ldots z_n$ are the singular points of $f(z)$.

$\{\text{Res } f(z)\}_{z=z_i}$ = the coef of $\frac{1}{z-z_i}$ in the Laurent's Series of $f(z)$ about $z=z_i$ (by def of residues)

$$= b_1 = \frac{1}{2\pi i}\int_{C_i} \frac{f(z)}{(z-z_i)^{1-1}}\,dz$$

$$\text{Since } b_n = \frac{1}{2\pi i}\int_{C_i} \frac{f(z)}{(z-z_i)^{1-n}}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_i} \frac{f(z)}{z-z_i}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_i} f(z)\,dz$$

$$\int_{c_i} f(z)\,dz = 2\pi i \left\{ \text{Res } f(z) \right\}_{z=z_i} \rightarrow (2)$$

From (1) & (2),

$$\therefore \int_c f(z)\,dz = 2\pi i \left\{ \text{Res } f(z) \right\}_{z=z_1} + 2\pi i \left\{ \text{Res } f(z) \right\}_{z=z_2} + \cdots + 2\pi i \left\{ \text{Res } f(z) \right\}_{z=z_n}$$

$$= 2\pi i \left\{ \left[\text{Res } f(z)\right]_{z=z_1} + \left[\text{Res } f(z)\right]_{z=z_2} + \cdots + \left[\text{Res } f(z)\right]_{z=z_n} \right\}$$

$$= 2\pi i \left\{ \text{Sum of residues of } f(z) \text{ at } z = z_1, z_2, \ldots z_n \right\}$$

$$= 2\pi i \sum_i R_i$$

Problems:

1) Evaluate $\int_c \frac{e^z}{(z+2)(z+1)^2}\,dz$ where c is the circle $|z|=3$.

Sol: Let $f(z) = \frac{e^z}{(z+2)(z+1)^2}$

The Poles of $f(z)$ are given by

$(z+2)(z+1)^2 = 0$

$z=-1$ is a pole of order 2.

$z=-2$ is a pole of order 1.

Given: $|z|=3$

$z=-1 \Rightarrow |z| = 1 < 3$, lies inside c.

$z=-2 \Rightarrow |z| = 2 < 3$, lies inside by c.

$$\{\operatorname{Res} f(z)\}_{z=-2} = \lim_{z\to -2} \cancel{(z+2)} \frac{e^z}{\cancel{(z+2)}(z+1)^2}$$

$$= \frac{e^2}{(-2+1)^2}$$

$$= e^2$$

$$\{\operatorname{Res} f(z)\}_{z=-1} = \lim_{z\to -1} \frac{1}{1!} \frac{d}{dz} \left((z+1)^2 \frac{e^z}{(z+1)^2(z+2)} \right)$$

$$= \lim_{z\to -1} \frac{d}{dz} \frac{e^2}{(z+2)}$$

$$= \lim_{z\to -1} \frac{-e^2}{(z+2)^2} = \frac{-e^2}{(-1+2)^2} = -e^2$$

By Cauchy's residue theorem,

$$\int_c f(z)\,dz = 2\pi i \left(\text{sum of the residues of } f(z) \text{ at the poles which lie inside } c \right)$$

$$\int_c \frac{e^z\,dz}{(z+2)(z+1)^2} = 2\pi i (e^2 - e^2)$$

$$= 0$$

$$\boxed{\int_c \frac{e^z\,dz}{(z+2)(z+1)^2} = 0}$$

2) Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$ where c is the circle $|z| = \frac{3}{2}$.

Sol: Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

The Poles of f(z) are

$z(z-1)(z-2) = 0$

$z = 0, z = 1, z = 2$ are Poles of order 1.

$z = 0 \Rightarrow |z| = 0 < \frac{3}{2}$ lies inside c.

$z = 1 \Rightarrow |z| = 1 < \frac{3}{2}$ lies inside c.

$z = 2 \Rightarrow |z| = 2 > \frac{3}{2}$ lies outside c.

$\{Res\ f(z)\}_{z=0} = \lim_{z \to 0} (z-0) \frac{4-3z}{z(z-1)(z-2)}$

$= \frac{4}{(-1)(-2)} = 2$

$\{Res\ f(z)\}_{z=1} = \lim_{z \to 1} (z-1) \frac{4-3z}{z(z-1)(z-2)}$

$= \frac{1}{-1} = -1$

$\{Res\ f(z)\}_{z=2} = 0$ (lies outside c)

By Cauchy's residue theorem,

$\int_C f(z) dz = 2\pi i$ (Sum of residues of f(z) at the poles which lie inside c)

$= 2\pi i (2-1) = 2\pi i$