Laplace transforms of derivatives :

If $L[f(t)] = F(s)$ then $L[f'(t)] = sF(s) - f(0)$.

Proof :

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

Integrating by Parts, we get

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\,(-se^{-st})\,dt$$

$$= \left[e^{-\infty} f(\infty) - e^{0} f(0)\right] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sL\{f(t)\}$$

$$= sF(s) - f(0)$$

$$\boxed{L[f'(t)] = sF(s) - f(0)}$$

Corollary :

Let $f''(t) = s^2F(s) - sf(0) - f'(0)$

Let $L[g'(t)] = sG(s) - g(0)$

we know that,

$$L[f'(t)] = sL[f(t)] - f(0)$$

Replace $f(t) \to f'(t)$ & $f'(t) \to f''(t)$ & $f(0) \to f'(0)$

$$L[f''(t)] = sL[f'(t)] - f'(0)$$

$$= s\left[s(L(f(t)) - f(0)\right] - f'(0)$$

$$= s^2L[f(t)] - sf(0) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Laplace transform of integrals:

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t)\,dt$ and $g(0) = 0$ then $g'(t) = f(t)$

We know that

$$L[g'(t)] = sL(g(t) - g(0))$$
$$= sL(g(t))$$
$$L[g(t)] = \frac{1}{s} L[g'(t)]$$
$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)]$$
$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Derivative of Laplace Transform (or) Laplace transform of $t f(t)$:

If $L[f(t)] = F(s)$ then

$$L[t f(t)] = -\frac{d}{ds} F(s)$$

Proof:

We know that,

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st} t f(t)\,dt$$

$$= -L[t f(t)]$$

$$L[t f(t)] = -\frac{d}{ds}[F(s)].$$

In general,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)].$$

Laplace Transforms of Derivatives:

1) Find $L[t \sin at]$

Sol:. $f(t) = t \sin at$

$$f'(t) = at\cos at + \sin at$$

$$f''(t) = a[-at\sin at + \cos at] + a\cos at$$

$$= 2a\cos at - a^2 t\sin at$$

$$f(0) = 0,\ f'(0) = 0$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$L[2a\cos at - a^2 t\sin at] = s^2 L[t\sin at] - s(0) - 0$$

$$2aL(\cos at) - a^2 L(t\sin at) = s^2 L(t\sin at)$$

$$(s^2+a^2)L(t\sin at) = 2aL(\cos at)$$

$$(s^2+a^2)L(t\sin at) = 2a \cdot \frac{s}{a^2+s^2}$$

$$L(t\sin at) = \frac{2as}{(s^2+a^2)^2}$$

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1) Find $L[t\cos at]$

Sol:- $L[t\,f(t)] = -\frac{d}{ds}[L(f(t))]$

$$L[t\cos at] = -\frac{d}{ds}[L(\cos at)]$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\left\{\frac{s^2+a^2-s(2s)}{(s^2+a^2)^2}\right\}$$

$$= -\left\{\frac{a^2-s^2}{(s^2+a^2)^2}\right\}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

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2) Find $L[te^{2t}\sin 3t]$

Sol:- $L[te^{2t}\sin 3t] = -\frac{d}{ds}\cdot\{L(e^{2t}\sin 3t)\}$

$$= \frac{-d}{ds}\left\{L(\sin 3t)\right\}_{s\to s-2}$$

$$= \frac{-d}{ds}\left\{\left(\frac{3}{s^2+9}\right)_{s\to s-2}\right\}$$

$$= -\left\{\frac{-3(2s)}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \left\{\frac{6s}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \frac{6(s-2)}{((s-2)^2+9)^2}$$

$$L[te^{2t}\sin 3t] = \frac{6(s-2)}{(s^2-4s+13)^2}$$

3) Find $L\left[t^2 e^{-2t}\cos t\right]$

Sol: $L\left[t^2 e^{-2t}\cos t\right] = (-1)^2 \frac{d^2}{ds^2}\left\{L(e^{-2t}\cos t)\right\}$

$$= \frac{d^2}{ds^2}\left\{L(\cos t)_{s\to s+2}\right\}$$

$$= \frac{d^2}{ds^2}\left\{\frac{s}{s^2+1}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{s^2+1-s(2s)}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+1)^2(-2s)-(1-s^2)\,2(s^2+1)(2s)}{(s^2+1)^4}\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+1)(-2s)-4s(1-s^2)}{(s^2+1)^3}\right\}_{s\to s+2}$$

$$= \frac{[(s+2)^2+1]\,[-2(s+2)]-4(s+2)[1-(s+2)^2]}{((s+2)^2+1)^3}$$

$$= \frac{(s^2+4s+5)(-2s-4)+(4s+8)(s^2+4s+3)}{((s+2)^2+1)^3}$$

$$= \frac{2s^3+12s^2+18s+4}{(s^2+4s+5)^3}$$

$$L\left[t^2e^{-2t}\cos t\right] = \frac{2s^3+12s^2+18s+4}{(s^2+4s+5)^3}$$

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4) Find $L\left[\frac{\sin 3t}{t}\right]$

Sol: $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds = \int_s^\infty L[f(t)]\,ds$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty L(\sin 3t)\,ds$$

$$= \int_s^\infty \left(\frac{3}{s^2+9}\right)ds$$

$$= \int_s^\infty \frac{3}{s^2+3^2}\, ds$$

$$= 3 \cdot \frac{1}{3}\left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$$

$$= \cot^{-1}(s/3)$$

Integral of Laplace Transform (or) Laplace transform of $\frac{f(t)}{t}$:

If $L[f(t)] = F(s)$ and if $\lim_{t\to 0} \frac{f(t)}{t}$ exists then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

Proof:

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Integrating w.r.to 's' from s to ∞, we get

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,ds\right] dt$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-s}\right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$$

Problems:

1) Find $L\left(\frac{1-\cos t}{t}\right)$

Sol: $L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty L(1-\cos t)\, ds$

$$= \int_s^\infty \{L(1) - L(\cos t)\} ds$$

$$= \int_s^\infty \left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} ds$$

$$= \left(\log s - \frac{1}{2}\log(s^2+1)\right)_s^\infty$$

$$= \left(\log s - \log(s^2+1)^{1/2}\right)_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= \left[\log \frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \log 1 - \log \left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log \left[\frac{s}{\sqrt{s^2+1}}\right]^{-1}$$

$$= \log \left(\frac{\sqrt{s^2+1}}{s}\right)$$

$$L\left(\frac{1-\cos t}{t}\right) = \log\left(\frac{\sqrt{s^2+1}}{s}\right)$$

2) Find $L\left(\frac{e^{-3t}-e^{-4t}}{t}\right)$

Sol: $L(e^{-3t}-e^{-4t}) = \frac{1}{s+3} - \frac{1}{s+4}$

$$L\left(\frac{e^{-3t}-e^{-4t}}{t}\right) = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \left[\log(s+3) - \log(s+4)\right]_s^\infty$$

$$= \left(\log\left(\frac{s+3}{s+4}\right)\right)_s^\infty$$

$$= \left[\log\left(\frac{s+3}{s+4}\right)\right]_s^\infty$$

$$= \log\left(\frac{s+4}{s+3}\right).$$

3) Find $L\left(\frac{1-\cos at}{t}\right)$

Sol: $L\left(\frac{1-\cos at}{t}\right) = \int_s^\infty L(1-\cos at)\,ds$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{a^2+s^2}\right) ds$$

$$= \left(\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$= \left(\log\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right)_s^\infty$$

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

$$L\left(\frac{1-\cos at}{t}\right) = \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

4) Find $L\left(\frac{\cos at - \cos bt}{t}\right)$

Sol: $L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty L(\cos at - \cos bt)\,ds$

$$= \int_s^\infty \left(\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log \frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

5) Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$

Sol: $L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1}$

$$\left(\because L\int_0^t f(t)\,dt = \frac{1}{s}L[f(t)]\right)$$

$$= \left[\frac{1}{s}L(t\cos t)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{-d}{ds}L(\cos t)\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to s+1}$$

$$= \cancel{\left(s^2+1\right.} = \left[\frac{-1}{s}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s \to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

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6) Evaluate using Laplace transform $\int_0^{\infty} te^{-2t}\sin 3t\,dt$

Sol:- $\int_0^{t} te^{-2t}\sin 3t\,dt = \int_0^{\infty} e^{-2t}(t\sin 3t)\,dt$

$$= \left[\int_0^{\infty} e^{-st}(t\sin 3t)\,dt\right]_{s=2}$$

$$= \left[L(t\sin 3t)\right]_{s=2}$$

$$= \left[\frac{-d}{ds}L(\sin 3t)\right]_{s=2}$$

$$= \left(\frac{-d}{ds}\left(\frac{3}{s^2+9}\right)\right)_{s=2}$$

$$= \left(\frac{6s}{(s^2+9)^2}\right)_{s=2}$$

$$= \frac{12}{169}$$