Proof:-

$$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$L[u(t-a)] = \frac{e^{-as}}{s} \quad (s>0)$$

Transforms of Periodic functions:

A function $f(x)$ is said to be Periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the Period of the function.

The Laplace transformation of a periodic function $f(t)$ with Period p given by,

$$\boxed{L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt}$$

Problems:

1) Find the Laplace transform of the rectangular wave given by $f(t)=\begin{cases}1 & ,0<t<b\\ -1 & ,b<t<2b.\end{cases}$

Sol: Given: $f(t)=\begin{cases}1 & ,0<t<b\\ -1 & ,b<t<2b\end{cases}$

$$L[f(t)]=\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt.$$

This function is Periodic in the interval $(0,2b)$ with Period $2b$.

$$L[f(t)]=\frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st}f(t)\,dt$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}\,dt+\int_b^{2b} e^{-st}(-1)\,dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b-\left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\frac{-1}{s}\left(e^{-st}\right)_0^b+\frac{1}{s}\left(e^{-st}\right)_b^{2b}\right]$$

$$=\frac{1}{s(1-e^{-2bs})}\left[-\left(e^{-bs}-1\right)+\left(e^{-2bs}-e^{-bs}\right)\right]$$

$$=\frac{-e^{-bs}+1+\left(e^{-bs}\right)^2-e^{-bs}}{s\left(1-e^{-2bs}\right)}$$

$$= \frac{1-2e^{-bs}+e^{-bs}}{s(1-e^{-2bs})}$$

$$= \frac{1}{s(1-e^{-bs})(1+e^{-bs})}(1-e^{-bs})^2$$

$$= \frac{1}{s}\left(\frac{1-e^{-bs}}{1+e^{-bs}}\right)$$

$$= \frac{1}{s}\left(\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}\right)$$

$$= \frac{1}{s}\tanh\left(\frac{bs}{2}\right).$$

2) Find the Laplace transform of the half wave rectifies function $f(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$

Sol: $L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin \omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}\cdot\omega+\omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left(1+e^{-s\pi/\omega}\right)}{\left(1-e^{-s\pi/\omega}\right)\left(1+e^{-s\pi/\omega}\right)\left(s^2+\omega^2\right)}$$

$$= \frac{\omega}{\left(1-e^{-s\pi/\omega}\right)\left(s^2+\omega^2\right)}$$

3) Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases} \quad \text{with } f(t+2a) = f(t).$$

Sol: $L[f(t)] = \dfrac{1}{1-e^{-2as}} \displaystyle\int_0^{2a} e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[t\left(\frac{e^{-st}}{-s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[-t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^a + \left[-(2a-t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\left(-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2}\right) - \left(\frac{-1}{s^2}\right)\right] + \left[\left(\frac{e^{-2as}}{s^2}\right) - \left(\frac{-ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right)\right]\right\}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right]$$

$$= \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

Inverse Laplace Transform:

If the Laplace transform of $f(t)$ is $F(s)$ then $L[f(t)] = F(s)$. Then $f(t)$ is called an inverse Laplace transform of $F(s)$ and is written as $f(t) = L^{-1}[F(s)]$ where L^{-1} is called the inverse Laplace transform operator.

Table of Inverse Laplace Transforms:

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
1. $L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1$
2. $L(t) = \frac{1}{s^2}$	$L^{-1}\left(\frac{1}{s^2}\right) = t$
3. $L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$
4. $L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$