$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right]$$

$$= \frac{(1-e^{-as})^{\cancel{2}}}{s^2(1+e^{-as})\cancel{(1-e^{-as})}}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

Inverse Laplace Transform:

If the Laplace transform of $f(t)$ is $F(s)$ then $L[f(t)] = F(s)$. Then $f(t)$ is called an inverse Laplace transform of $F(s)$ and is written as $f(t) = L^{-1}[F(s)]$ where L^{-1} is called the inverse Laplace transform operator.

Table of Inverse Laplace Transforms:

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
1. $L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1$
2. $L(t) = \frac{1}{s^2}$	$L^{-1}\left(\frac{1}{s^2}\right) = t$
3. $L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$
4. $L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

5. $L(e^{-at}) = \frac{1}{s+a}$ $\qquad L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

6. $L(\sin at) = \frac{a}{s^2+a^2}$ $\qquad L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$

7. $L(\cos at) = \frac{s}{s^2+a^2}$ $\qquad L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

8. $L\left(\frac{\sin at}{a}\right) = \frac{1}{s^2+a^2}$ $\qquad L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

9. $L(\sinh at) = \frac{a}{s^2-a^2}$ $\qquad L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$

10. $L(\cosh at) = \frac{s}{s^2-a^2}$ $\qquad L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

11. $L(\delta(t)) = 1$ $\qquad L^{-1}(1) = \delta(t)$.

Important Results in Laplace Transforms:

If a and b are constants while $F(s)$ and $G(s)$ are the Laplace transform of $f(t)$ & $g(t)$ respectively.

1. **Linearity Property:**

$$L^{-1}[aF(s) + bG(s)] = aL^{-1}[F(s)] + bL^{-1}[G(s)]$$

2. **First shifting Property:**

(i) $L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]$

(ii) $L^{-1}[F(s-a)] = e^{at}L^{-1}[F(s)]$

3. Second shifting Property :.

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[e^{-as}F(s)] = g(t)$ where

$$g(t) = \begin{cases} f(t-a) & , t > a \\ 0 & , t < a \end{cases}$$

4. Change of scale Property ::

$$L^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right)$$

5. Multiplication by s :.

If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$ then.

$$L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$$

Note :-

If $f(0) \neq 0$ then $L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)] + f(0)\delta(t)$.

Problem Identification :.

If $L^{-1}\left[\frac{s}{\text{quadratic equation}}\right]$ then use result 5.

If $L^{-1}\left[\frac{s}{\text{Linear equation}}\right]$ then we use the above note.

6. Division by s :.

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

7. Inverse Laplace transform of derivatives : .

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F'(s)] = -t L^{-1}[F(s)]$

Problem Identification:

If $L^{-1}\left[\frac{s + \text{any term}}{(\text{Quadratic equ})^2}\right]$ then we use the above result.

8. Note:

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F(s)] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}F(s)\right]$

Problem Identification:

If L^{-1} [log function or cot function or tan function] then we use the above result.

Problems:

1) Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Sol: $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s^2+a^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$

$= L^{-1}\left[\frac{1}{s^2+b^2} - \frac{a^2}{(s^2+a^2)(s^2+b^2)}\right]$

$= L^{-1}\left[\frac{1}{s^2+b^2}\right] - a^2 L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$

$= \frac{1}{b} L^{-1}\left(\frac{b}{s^2+b^2}\right) - \frac{a^2}{b^2-a^2} L^{-1}\left(\frac{b^2-a^2}{(s^2+a^2)(s^2+b^2)}\right)$

$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2} L^{-1}\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right)$

$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$

2) Find $L^{-1}\left(\frac{2s-5}{9s^2-25}\right)$

Sol: $L^{-1}\left(\frac{2s-5}{9s^2-25}\right) = L^{-1}\left(\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right)$

$$= L^{-1}\left(\frac{2s}{9\left(s^2-\left(\frac{5}{3}\right)^2\right)} - \frac{5}{9\left(s^2-\left(\frac{5}{3}\right)^2\right)}\right)$$

$$= \frac{2}{9} L^{-1}\left[\frac{s}{s^2-\left(\frac{5}{3}\right)^2}\right] - \frac{1}{3} L^{-1}\left[\frac{5/3}{s^2-(5/3)^2}\right]$$

$$= \frac{2}{9}\cosh\frac{5}{3}t - \frac{1}{3}\sinh\frac{5}{3}t$$

3) Find $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$

Sol: $L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = \frac{d}{dt}\left[L^{-1}\left(\frac{1}{(s+2)^2+4}\right)\right]$

$$= \frac{d}{dt}\left[e^{-2t} L^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$$

$$= \frac{d}{dt}\left[\frac{e^{-2t}}{2} L^{-1}\left(\frac{2}{s^2+2^2}\right)\right]$$

$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}\sin 2t\right)$$

$$= \frac{1}{2}\frac{d}{dt}\left(e^{-2t}\sin 2t\right)$$

$$= \frac{1}{2}\left[e^{-2t}\cos 2t(2) + \sin 2t(-2)e^{-2t}\right]$$

$$= \frac{1}{2}\left[2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t\right]$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$

4) Find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Sol:- $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\log \frac{s+1}{s-1}\right)\right]$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\log(s+1) - \log(s-1)\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{-1}{t}\left(e^{-t} - e^{t}\right)$$

$$= \frac{e^{t} - e^{-t}}{2}$$

$$= \frac{2}{t}\left[\frac{e^{t} - e^{-t}}{2}\right]$$

$$= \frac{2}{t} \sinh t$$

Partial fractions:-

1) Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Sol:- $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

$s = -1 \Rightarrow 1 = A(0) + B(-1) + C(0)$

$$\boxed{B = -1}$$

$S = -2 \Rightarrow 1 = A(0) + B(0) + C(-2)(-1)$

$$1 = 2C$$

$\boxed{C=\frac{1}{2}}$

Put $s=0 \Rightarrow 1 = A(1)(2)+B(0)+C(0)$

$1 = 2A$

$\boxed{A=\frac{1}{2}}$

$$\frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{2}1 - e^{-t} + \frac{1}{2}e^{-2t}$$

$$= \frac{1}{2}\left[1-2e^{-t}+e^{-2t}\right]$$

---

2) Find $L^{-1}\left[\frac{s^2}{(s+1)(s^2+4)}\right]$

Sol: $$\frac{s^2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$s^2 = A(s^2+4)+(Bs+C)(s+1)$$

$s=-1 \Rightarrow 1 = A(5)+B(-1)(0)+C(0)$

$\boxed{A=\frac{1}{5}}$

$s=0 \Rightarrow 0 = 4A+C$

$\boxed{C=-\frac{4}{5}}$

$s=-4 \Rightarrow 16 = 20A+12B-3C$

$\boxed{B=4/5}$

$$L^{-1}\left[\frac{s^2}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{1/5}{s+1} + \frac{4/5\, s - 4/5}{s^2+4}\right]$$

$$= \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right) + \frac{4}{5}L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{4}{5}L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{1}{5}e^{-t} + \frac{4}{5}\cos 2t - \frac{4}{5}\,\frac{\sin 2t}{2}$$

---

3) Find $L^{-1}\left(\frac{s^2+2s+1}{(s+3)(s-3)(s+1)}\right)$

Sol: $\frac{1}{3}e^{-3t} + \frac{2}{3}e^{3t}$

4) Find $L^{-1}\left(\frac{s}{(s-3)(s^2+4)}\right)$

Sol: $\frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{3}\sin t$

Convolution:

Definition:

If $f(t)$ and $g(t)$ are two functions defined for $t \geq 0$ then the Convolution of $f(t)$ and $g(t)$ is defined as

$$f(t) + g(t) = (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

Note: $f(t) * g(t) = g(t) * f(t)$.

Convolution theorem:

If $f(t)$ and $g(t)$ are two laplace transformable functions defined for $t \geq 0$ then $L[f(t) * g(t)]$ is given by