Laurent's Series

Let C_1 and C_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

Let $f(z)$ be analytic inside and on the annular region R between C_1 and C_2. Then for any $z \in R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

Problems :

(1) Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in a Laurent's series if (i) $|z| < 2$ (ii) $|z| > 3$ and (iii) $2 < |z| < 3$.

Soln: Using partial fractions,

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3} \rightarrow (1)$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A(z+2)(z+3) + B(z+3) + C(z+2)}{(z+2)(z+3)}$$

$$z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2)$$

Put $z = -2$

$$(-2)^2 - 1 = A(0) + B(-2+3) + 0$$

$$4 - 1 = B$$

$$\boxed{B = 3}$$

Put $z = -3$

$$(-3)^2 - 1 = 0 + 0 + C(-3+2)$$

$$9 - 1 = -C$$

$$\boxed{C = -8}$$

Put $z = 0$

$$0 - 1 = A(2)(3) + B(3) + C(2)$$

$$-1 = 6A + 3B + 2C$$

$$= 6A + 3(3) + 2(-8)$$

$$= 6A + 9 - 16$$

$$-1 = 6A - 7$$

$$6A = 7 - 1 = 6$$

$$\boxed{A = 1}$$

$$(1) \Rightarrow f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} \rightarrow (2)$$

(i) $|z| < 2$

$$\Rightarrow \frac{|z|}{2} < 1$$

(2) $\Rightarrow f(z) = 1 + \dfrac{3}{2\left(1+\frac{z}{2}\right)} - \dfrac{8}{3\left(1+\frac{z}{3}\right)}$

$= 1 + \dfrac{3}{2}\left(1+\dfrac{z}{2}\right)^{-1} - \dfrac{8}{3}\left(1+\dfrac{z}{3}\right)^{-1}$

$= 1 + \dfrac{3}{2}\left[1 - \dfrac{z}{2} + \left(\dfrac{z}{2}\right)^2 - \cdots\right]$

$- \dfrac{8}{3}\left[1 - \dfrac{z}{3} + \left(\dfrac{z}{3}\right)^2 + \cdots\right]$

Formula

$\boxed{\because (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots}$

(ii) $|z| > 3$

$3 < |z|$

$\dfrac{3}{|z|} < 1$.

(2) $\Rightarrow f(z) = 1 + \dfrac{3}{z\left(1+\frac{2}{z}\right)} - \dfrac{8}{z\left(1+\frac{3}{z}\right)}$

$= 1 + \dfrac{3}{z}\left(1+\dfrac{2}{z}\right)^{-1} - \dfrac{8}{z}\left(1+\dfrac{3}{z}\right)^{-1}$

$= 1 + \dfrac{3}{z}\left[1 - \dfrac{2}{z} + \left(\dfrac{2}{z}\right)^2 - \cdots\right]$

$- \dfrac{8}{z}\left[1 - \dfrac{3}{z} + \left(\dfrac{3}{z}\right)^2 - \cdots\right]$

(iii) $2 < |z| < 3$

$2 < |z|$ and $|z| < 3$

$\dfrac{2}{|z|} < 1$ and $\dfrac{|z|}{3} < 1$

$$② \Rightarrow 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\right]$$

$$-\frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right]$$

Soln:

Given: $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

When $z=0$, $A=1$

$z=-1$, $C=-3$

$z=2$, $B=2$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

(i) $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{-2(1-\frac{z}{2})} - \frac{3}{z+1}$$

$$= \frac{1}{z} - \left(1-\frac{z}{2}\right)^{-1} - 3(z+1)^{-1}$$

$$= \frac{1}{z} - \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right] - 3\left[1 - z + z^2 - z^3 + \cdots\right]$$

(ii) $|z| > 3 \Rightarrow \frac{3}{|z|} < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z(1-\frac{2}{z})} - \frac{3}{z(1+\frac{1}{z})}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{3}{z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right]$$

(iii) $2 < |z| < 3$

$|z| > 2 \quad , \quad |z| < 3$

$\Rightarrow \frac{2}{|z|} < 1 \quad , \quad \frac{|z|}{3} < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z\left(1-\frac{2}{z}\right)} - \frac{3}{z\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right) - \frac{3}{z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\cdots\right]$$

(iv) $1 < |z+1| < 3$

Let $u = z+1 \Rightarrow z = u-1$

$1 < |u| < 3$

$\Rightarrow \frac{1}{|u|} < 1 \quad , \quad \left|\frac{u}{3}\right| < 1$

$$f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u\left[1-\frac{1}{u}\right]} + \frac{2}{(-3)\left(1-\frac{u}{3}\right)} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\cdots\right) - \frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots\right) - 3/u$$

$$f(z) = \frac{1}{z+1}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\cdots\right]$$
$$- \frac{2}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right) - \frac{3}{z+1}$$