

**DEPARTMENT OF MATHEMATICS**Laurent's Series

Let C_1 and C_2 be two concentric circles
 $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

Let $f(z)$ be analytic inside and on the
 annular region R between C_1 and C_2 . Then for any
 $z \in R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$\text{Where } a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

Problems :

① Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in a Laurent's

series if (i) $|z| < 2$ (ii) $|z| > 3$ and
 (iii) $2 < |z| < 3$.

Soln: Using partial fractions,

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3} \rightarrow \text{①}$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A(z+2)(z+3) + B(z+3) + C(z+2)}{(z+2)(z+3)}$$

$$z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2)$$



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) & amp;

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & amp; B.Tech.IT)

COIMBATORE-641 035, TAMIL NADU

DEPATMENT OF MATHEMATICS

$$\text{Put } z = -2$$

$$(-2)^2 - 1 = A(0) + B(-2+3) + 0$$

$$4 - 1 = B$$

$$\boxed{B = 3}$$

$$\text{Put } z = -3$$

$$(-3)^2 - 1 = 0 + 0 + C(-3+2)$$

$$9 - 1 = -C$$

$$\boxed{C = -8}$$

$$\text{put } z = 0$$

$$0 - 1 = A(2)(3) + B(3) + C(2)$$

$$-1 = 6A + 3B + 2C$$

$$= 6A + 3(3) + 2(-8)$$

$$= 6A + 9 - 16$$

$$-1 = 6A - 7$$

$$6A = 7 - 1 = 6$$

$$\boxed{A = 1}$$

$$\textcircled{1} \Rightarrow f(z) = 1 + \frac{3}{z+2} + \frac{8}{z+3} \rightarrow \textcircled{2}$$

$$\textcircled{1} \Rightarrow (i) |z| < 2$$

$$\Rightarrow \frac{|z|}{2} < 1$$



DEPARTMENT OF MATHEMATICS

$$(2) \Rightarrow f(z) = 1 + \frac{3}{2 \left(1 + \frac{z}{2}\right)} - \frac{8}{3 \left(1 + \frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2} \left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2} \left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \dots\right]$$

$$- \frac{8}{3} \left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \dots\right]$$

Formula

$$\because (1+z)^{-1} = 1 - z + z^2 - z^3 + \dots$$

(ii) $|z| > 3$

$$3 < |z|$$

$$\frac{3}{|z|} < 1$$

$$(2) \Rightarrow f(z) = 1 + \frac{3}{z \left(1 + \frac{2}{z}\right)} - \frac{8}{z \left(1 + \frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z} \left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z} \left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z} \left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \dots\right]$$

$$- \frac{8}{z} \left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \dots\right]$$

(iii) $2 < |z| < 3$

$$2 < |z| \text{ and } |z| < 3$$

$$\frac{2}{|z|} < 1 \text{ and } \frac{|z|}{3} < 1$$



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)

COIMBATORE-641 035, TAMIL NADU



DEPARTMENT OF MATHEMATICS

$$\begin{aligned}
 (2) \Rightarrow 1 + \frac{-3}{2 \left(1 + \frac{z}{2}\right)} - \frac{8}{3 \left(1 + \frac{z}{3}\right)} &= \dots \\
 &= 1 + \frac{3}{2} \left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1} \\
 &= 1 + \frac{3}{2} \left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots\right] \\
 &\quad - \frac{8}{3} \left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \dots\right]
 \end{aligned}$$



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)

COIMBATORE-641 035, TAMIL NADU



DEPARTMENT OF MATHEMATICS

Soln:

Given: $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

When $z=0$, $A = 1$

$z=-1$, $C = -3$

$z=2$, $B = 2$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

(i) $|z| < 2 \Rightarrow \left| \frac{z}{2} \right| < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{-2(1-\frac{z}{2})} - \frac{3}{z+1}$$

$$= \frac{1}{z} - \left(1-\frac{z}{2}\right)^{-1} - 3(z+1)^{-1}$$

$$= \frac{1}{z} - \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right] - 3\left[1 - z + z^2 - z^3 + \dots\right]$$

(ii) $|z| > 3 \Rightarrow \frac{3}{|z|} < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \left(\frac{1}{z} + \frac{2}{z(1-\frac{2}{z})}\right) - \frac{3}{z(1+\frac{1}{z})}$$

$$= \frac{1}{z} + \frac{2}{z} \left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z} \left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z} \left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots\right] - \frac{3}{z} \left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \dots\right]$$

$$(iii) \quad 2 < |z| < 3$$

$$|z| > 2, \quad |z| < 3$$

$$\Rightarrow \frac{2}{|z|} < 1, \quad \frac{|z|}{3} < 1$$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z(1-\frac{2}{z})} - \frac{3}{z(1+\frac{1}{z})}$$

$$= \frac{1}{z} + \frac{2}{z} \left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \dots \right) - \frac{3}{z} \left(1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \dots \right)$$

$$(iv) \quad 1 < |z+1| < 3$$

$$\text{Let } u = z+1 \Rightarrow z = u-1$$

$$1 < |u| < 3$$

$$\Rightarrow \frac{1}{|u|} < 1, \quad \left| \frac{u}{3} \right| < 1$$

$$f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u \left[1 - \frac{1}{u} \right]} + \frac{2}{(-3) \left(1 - \frac{u}{3} \right)} - \frac{3}{u}$$

$$= \frac{1}{u} \left(1 - \frac{1}{u} \right)^{-1} - \frac{2}{3} \left(1 - \frac{u}{3} \right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u} \left(1 + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \dots \right) - \frac{2}{3} \left(1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \dots \right) - \frac{3}{u}$$

$$f(z) = \frac{1}{z+1} \left[1 + \left(\frac{1}{z+1}\right) + \left(\frac{1}{z+1}\right)^2 + \dots \right]$$

$$- \frac{2}{3} \left(1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \dots \right) - \frac{3}{z+1}$$