



DEPARTMENT OF MATHEMATICS

Transforms of periodic functions:

A function $f(x)$ is said to be periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x . The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace transformation of a periodic function $f(t)$ with period p given by,

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt.$$

Problems:

- (1) Find the Laplace transform of the rectangular wave given by,

$$f(t) = \begin{cases} 1, & 0 \leq t < b \\ -1, & b \leq t < 2b \end{cases}$$

Soln:

$$\text{Given: } f(t) = \begin{cases} 1, & 0 \leq t < b \\ -1, & b \leq t < 2b \end{cases}$$

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt$$

This fn is periodic in the interval $(0, 2b)$ with

$$\begin{aligned} \mathcal{L}[f(t)] &= \frac{1}{1 - e^{-2bs}} \int_0^{2b} e^{-st} f(t) dt \quad \text{Period } 2b. \\ &= \frac{1}{1 - e^{-2bs}} \left[\int_0^b e^{-st} dt + \int_b^{2b} e^{-st} (-1) dt \right] \end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{1-e^{-2bs}} \left[\left(\frac{e^{-st}}{-s} \right)_0^b - \left(\frac{e^{-st}}{-s} \right)_b^{2b} \right] \\
 &= \frac{1}{1-e^{-2bs}} \left[-\frac{1}{s} (e^{-st})_0^b + \frac{1}{s} (e^{-st})_b^{2b} \right] \\
 &= \frac{1}{s(1-e^{-2bs})} \left[-(e^{-bs}-1) + (e^{-2bs}-e^{-bs}) \right] \\
 &= \frac{-e^{-bs}+1+(e^{-bs})^2-e^{-bs}}{s(1-e^{-2bs})} = \frac{1-2e^{-bs}+(e^{-bs})^2}{s(1-e^{-2bs})} \\
 &= \frac{1}{s(1-e^{-bs})(1+e^{-bs})} (1-e^{-bs})^2 \\
 &= \frac{1}{s} \left(\frac{1-e^{-bs}}{1+e^{-bs}} \right) \\
 &= \frac{1}{s} \left(\frac{e^{sb/2} - e^{-sb/2}}{e^{sb/2} + e^{-sb/2}} \right) \\
 &= \frac{1}{s} \tanh \left(\frac{bs}{2} \right)
 \end{aligned}$$

② Find the Laplace transform of the half wave rectifier function

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Soln:

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t) dt$$



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$$\begin{aligned}
 &= \left[\frac{1}{1 - e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + 0 \right] \right. \\
 &= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega} \\
 &= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega} \cdot \omega + \omega}{s^2 + \omega^2} \right] \\
 &= \frac{\omega [1 + e^{-s\pi/\omega}]}{(1 - e^{-s\pi/\omega})(1 + e^{-s\pi/\omega})(s^2 + \omega^2)} \\
 &= \frac{\omega}{(1 - e^{-s\pi/\omega})(s^2 + \omega^2)}
 \end{aligned}$$

③ Find the Laplace transform of

$$f(t) = \begin{cases} t & , 0 < t < a \\ 2a - t & , a < t < 2a \end{cases} \quad \text{with } f(t+2a) = f(t)$$

Soln :

$$\begin{aligned}
 \mathcal{L}[f(t)] &= \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) \, dt \\
 &= \frac{1}{1 - e^{-2as}} \left[\int_0^a e^{-st} t \, dt + \int_a^{2a} e^{-st} (2a - t) \, dt \right] \\
 &= \frac{1}{1 - e^{-2as}} \left\{ \left[t \left(\frac{e^{-st}}{-s} \right) - \left(\frac{e^{-st}}{s^2} \right) \right]_0^a + \left[(2a - t) \left(\frac{e^{-st}}{-s} \right) - (-1) \left(\frac{e^{-st}}{s^2} \right) \right]_a^{2a} \right\} \\
 &= \frac{1}{1 - e^{-2as}} \left\{ \left[-t \frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a - t) \frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right\}
 \end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{1 - e^{-2as}} \left\{ \left[\left(-a \frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) - \left(-\frac{1}{s^2} \right) \right] + \right. \\
 &\quad \left. \left[\left(\frac{e^{-2as}}{s^2} \right) - \left(-a \frac{e^{-as}}{s} + \frac{e^{-as}}{s^2} \right) \right] \right\} \\
 &= \frac{1}{1 - e^{-2as}} \left[\frac{-a e^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{a e^{-as}}{s} - \frac{e^{-as}}{s^2} \right] \\
 &= \frac{1}{1 - e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right] \\
 &= \frac{(1 - e^{-as})^2}{s^2 (1 + e^{-as}) (1 - e^{-as})} \\
 &= \frac{1 - e^{-as}}{s^2 (1 + e^{-as})} \\
 &= \frac{1}{s^2} \tanh \left(\frac{as}{2} \right)
 \end{aligned}$$