Initial value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(s)$ then

$$\lim_{t \to 0} [f(t)] = \lim_{s \to \infty} [sF(s)]$$

Proof:

We know that

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$= sF(s) - f(0)$$

$$\Rightarrow sF(s) = L[f'(t)] + f(0)$$

$$sF(s) = \int_0^\infty e^{-st} f'(t)\, dt + f(0).$$

Taking limit as $s \to \infty$ on both sides we get,

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \left\{ \int_0^\infty e^{-st} f'(t)\, dt + f(0) \right\}$$

$$= \lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty \lim_{s\to\infty} e^{-st} f'(t)\,dt + f(0)$$

$$= 0 + f(0)$$

$$= \lim_{t\to 0} f(t)$$

Hence $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$.

Final value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(s)$ then

$$\lim_{t\to\infty} [f(t)] = \lim_{s\to 0} [sF(s)]$$

Proof:

We know that,

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$= sF(s) - f(0)$$

$$\Rightarrow sF(s) = L[f'(t)] + f(0)$$

Taking limit $s \to 0$ on both sides, we get,

$$\lim_{s\to 0} [sF(s)] = \lim_{s\to 0} \left\{ \int_0^\infty e^{-st} f'(t)\,dt + f(0) \right\}$$

$$= \int_0^\infty \lim_{s\to 0} e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty f'(t)\,dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0) = \lim_{t\to\infty} f(t)$$

Hence $\lim_{t\to\infty} f(t) = \lim_{s\to 0} [sF(s)]$

(1) Verify the initial and final value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Soln:

$$F(s) = L[1 + e^{-t}\sin t + e^{-t}\cos t]$$

$$= L(1) + L(\sin t)_{s \to s+1} + L(\cos t)_{s \to s+1}$$

$$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s \to s+1} + \left(\frac{s}{s^2+1}\right)_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$$

$$\therefore sF(s) = s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

Initial value theorem:

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} [1 + e^{-t}(\sin t + \cos t)] = 1 + 1 = 2$$

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \lim_{s \to \infty} 1 + \frac{s^2+2s}{s^2+2s\left(1+\frac{2}{s^2+2s}\right)}$$

$$= 1 + 1$$

$$= 2$$

Hence $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) = 2$

IVT is verified.

Final value theorem:

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} [1 + e^{-t}(\sin t + \cos t)] = 1$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \lim_{s\to 0}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1$$

Hence $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 1$, FVT is Verified

Laplace Transform of Some Special functions:

Unit Step function:

The unit step function also called Heavisides unit function is defined as,

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

This is the unit step functions at $t=a$. It can also be denoted by $H(t-a)$ or $u_a(t)$.

Result:

Laplace Transform of unit step function is $\frac{e^{-as}}{s}$ i.e., $L[u(t-a)] = \frac{e^{-as}}{s}$

Proof:

$$L[u(t-a)] = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st}\, u(t-a)\, dt + \int_a^\infty e^{-st}\, u(t-a)$$

$$= 0 + \int_a^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{e^{-as}}{s} \quad (s > 0)$$