# UNIT-V

## LAPLACE TRANSFORMS

### INTRODUCTION:

Laplace Transformation, named after a great French Mathematician Pierre Simon De Laplace (1749 – 1827) who used such transformations in the "Theory of probability".

### Uses of Laplace Transformation:

1. It is used to find the solution of linear differential equations - ordinary as well as partial.
2. It helps in solving the differential equation with boundary values without finding the general solution and then finding the values of the arbitrary constants

### Transformation:

A transformation is an operation which converts a mathematical expression to a different but equivalent form.

### Laplace Transformation: Definition:

Let $f(t)$ be a function of $t$ defined for $t>0$. Then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ or $F(s)$ is defined by,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

Provided the integral exists.

## Conditions for existence of Laplace transform:

(i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a, b]$ where $a > 0$.

(ii) $f(t)$ should be of exponential order.

### Exponential order:

A function $f(t)$ is said to be of exponential order if,

$$\lim_{t \to \infty} e^{-st} f(t) = 0.$$

### Example:

1. $t^2$ is of exponential order.

$$\lim_{t \to \infty} e^{-st} f(t) = \lim_{t \to \infty} e^{-st} t^2$$

$$= \lim_{t \to \infty} \frac{t^2}{e^{+st}} \quad \left[\frac{\infty}{\infty} \text{ Indeterminate form}\right.$$

$$= \lim_{t \to \infty} \frac{2t}{se^{+st}} \quad \text{Apply L'Hospital rule}$$

$$= \lim_{t \to \infty} \frac{2}{s^2 e^{st}} = \frac{2}{\infty} = 0.$$

2. $e^{t^2}$ is not of exponential order

$$\lim_{t \to \infty} e^{-st} f(t) = \lim_{t \to \infty} e^{-st} e^{t^2}$$

$$= \lim_{t \to \infty} e^{-st + t^2}$$

$$= e^{\infty} = \infty$$

$\therefore e^{t^2}$ is not of exponential order.

## Transforms of elementary functions:

(1) $L(1) = \frac{1}{s}$ where $s > 0$.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(1) = \int_0^\infty e^{-st} \cdot 1\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}(0-1) = \frac{1}{s}$$

$$\boxed{L(1) = \frac{1}{s}}$$

(2) $L(K) = \frac{k}{s}$.

(3) $L(t) = \frac{1!}{s^2}$

$$L(t) = \int_0^\infty e^{-st} \cdot t\,dt$$

$$= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$

$$\boxed{L(t) = \frac{1!}{s^2}}$$

(4) $L(t^2) = \frac{2!}{s^3}$

(5) $L(t^n) = \frac{\Gamma_{n+1}}{s^{n+1}}$ if $s > 0$ & $n > -1$

$$L(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

put $x = st \Rightarrow dx = s\,dt$

$$\frac{dx}{s} = dt$$

$$L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \cdot \frac{x^n}{s^{n+1}} \, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$\boxed{L(t^n) = \frac{\Gamma_{n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$

(6) $L(e^{at}) = \frac{1}{s-a}$ if $s-a>0$.

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$\boxed{L(e^{at}) = \frac{1}{s-a} \text{ if } s-a>0}$$

(7) $L(e^{-at}) = \frac{1}{s+a}$ if $s+a>0$

$$L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} \, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$\boxed{L(e^{-at}) = \frac{1}{s+a} \text{ if } s+a>0}$$

(8) To find $L(\cos at)$ & $L(\sin at)$:

We know $e^{i\theta} = \cos\theta + i\sin\theta$.

$$L(e^{iat}) = \frac{1}{s-ia}$$

$$= \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2+a^2}$$

$$L(\cos at + i\sin at) = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Equating real & imaginary parts,

$$\boxed{L(\cos at) = \frac{s}{s^2+a^2}}$$

$$\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$$

(9) To find $L(\sinh at)$:

$$L[\sinh at] = L\left(\frac{e^{at}-e^{-at}}{2}\right)$$

$$= \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left(\frac{2a}{s^2-a^2}\right)$$

$$\boxed{L(\sinh at) = \frac{a}{s^2-a^2}} \text{ for } s^2 > a^2$$