



DEPARTMENT OF MATHEMATICS

First Shifting property:

If $\mathcal{L}\{f(t)\} = F(s)$ then -

$$(i) \mathcal{L}[e^{-at}f(t)] = \left\{ \mathcal{L}[f(t)] \right\}_{s \rightarrow s+a} = F(s+a)$$

$$(ii) \mathcal{L}[e^{at}f(t)] = \left\{ \mathcal{L}[f(t)] \right\}_{s \rightarrow s-a} = F(s-a)$$

Proof:

(i) We know that,

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$$\mathcal{L}[e^{-at}f(t)] = \int_0^{\infty} e^{-st} [e^{-at}f(t)] dt$$

$$= \int_0^{\infty} e^{-(s+a)t} f(t) dt$$

$$= F(s+a)$$

$$(ii) \mathcal{L}[e^{at}f(t)] = \int_0^{\infty} e^{-st} [e^{at}f(t)] dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t) dt$$

$$= F(s-a)$$

Second Shifting property:

If $\mathcal{L}\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t \geq a \\ 0, & t < a \end{cases}$

then $\mathcal{L}[g(t)] = e^{-as} F(s)$.

Proof:

$$\mathcal{L}[g(t)] = \int_0^{\infty} e^{-st} g(t) dt$$

$$= \int_a^{\infty} e^{-st} g(t) dt + \int_0^a e^{-st} g(t) dt$$



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$$\begin{aligned}
 \mathcal{L}[g(t)] &= 0 + \int_a^{\infty} e^{-st} f(t-a) dt \\
 &= \int_a^{\infty} e^{-st} f(t-a) dt \\
 \text{Put } t-a &= u \Rightarrow dt = du \\
 \text{When } t &= a \Rightarrow u = 0 \\
 t \rightarrow \infty &\Rightarrow u \rightarrow \infty \\
 \mathcal{L}[g(t)] &= \int_0^{\infty} e^{-s(u+a)} f(u) du \\
 &= \int_0^{\infty} e^{-us} e^{-as} f(u) du \\
 &= e^{-as} \int_0^{\infty} e^{-us} f(u) du \\
 &= e^{-as} \int_0^{\infty} e^{-st} f(t) dt \quad \text{Replace } u \rightarrow t \\
 \mathcal{L}[g(t)] &= e^{-as} F(s)
 \end{aligned}$$

Laplace transforms of derivatives:

If $\mathcal{L}[f(t)] = F(s)$ then

$$\mathcal{L}[f'(t)] = sF(s) - f(0).$$

Proof:

$$\mathcal{L}[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$$

Integrating by parts we get,

$$= \left[e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} f(t) (-se^{-st}) dt$$

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$$\begin{aligned}
 \mathcal{L}\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \\
 &= \int_0^{\infty} \frac{d}{dt} [e^{-st} f(t)] dt + s \int_0^{\infty} e^{-st} f(t) dt \\
 &= [e^{-st} f(t)]_0^{\infty} + s \int_0^{\infty} e^{-st} f(t) dt \\
 &= -f(0) + s \int_0^{\infty} e^{-st} f(t) dt \\
 &= s F(s) - f(0)
 \end{aligned}$$

Corollary:

$$\begin{aligned}
 \text{Let } f''(t) &= s^2 F(s) - s f(0) - f'(0) \\
 \text{Let } \mathcal{L}[g'(t)] &= s G(s) - g(0)
 \end{aligned}$$

We know that,

$$\mathcal{L}[f'(t)] = s \mathcal{L}[f(t)] - f(0)$$

Replace $f(t) \rightarrow f'(t)$ & $f'(t) \rightarrow f''(t)$ & $f(0) \rightarrow f'(0)$

$$\begin{aligned}
 \Rightarrow \mathcal{L}[f''(t)] &= s \mathcal{L}[f'(t)] - f'(0) \\
 &= s [s \mathcal{L}[f(t)] - f(0)] - f'(0) \\
 &= s^2 \mathcal{L}[f(t)] - s f(0) - f'(0) \\
 &= s^2 F(s) - s f(0) - f'(0)
 \end{aligned}$$

Laplace Transform of integrals:

If $\mathcal{L}[f(t)] = F(s)$ then $\mathcal{L}\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t) dt$ and $g(0) = 0$

then $g'(t) = f(t)$

WKT $\mathcal{L}[g'(t)] = s \mathcal{L}[g(t)] - g(0)$

$$\begin{aligned}
 &= s \mathcal{L}[g(t)] \\
 \Rightarrow \mathcal{L}[g(t)] &= \frac{1}{s} \mathcal{L}[g'(t)]
 \end{aligned}$$



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$$\Rightarrow \mathcal{L} \left[\int_0^t f(t) dt \right] = \frac{1}{s} \mathcal{L} [f(t)] \quad \left\{ \begin{array}{l} \because g(t) = \int_0^t f(t) dt \\ g'(t) = f(t) \end{array} \right.$$

$$\Rightarrow \mathcal{L} \left[\int_0^t f(t) dt \right] = \frac{F(s)}{s}$$

Derivative of Laplace Transform (or) Laplace transform of $t f(t)$:

If $\mathcal{L} [f(t)] = F(s)$ then

$$\mathcal{L} [t f(t)] = -\frac{d}{ds} F(s)$$

Proof:

We know that,

$$\mathcal{L} [f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^{\infty} \frac{\partial}{\partial s} (e^{-st}) f(t) dt$$

$$= \int_0^{\infty} -t e^{-st} f(t) dt$$

$$= - \int_0^{\infty} e^{-st} t f(t) dt$$

$$= - \mathcal{L} [t f(t)]$$

$$\Rightarrow \mathcal{L} [t f(t)] = -\frac{d}{ds} [F(s)]$$

In general,

$$\mathcal{L} [t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$



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Problems :

Change of Scale property :

- ① Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh ht] = \frac{1}{s^2 - h^2} = F(s)$$

$$\begin{aligned} L[\sinh 3t] &= \frac{1}{3} F\left(\frac{s}{3}\right) \\ &= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1} \\ &= \frac{1}{3} \left(\frac{9}{s^2 - 9} \right) \\ &= \frac{3}{s^2 - 9} \end{aligned}$$

- ② Find $L(\cos 5t)$ using change of scale property?

Soln:

$$L(\cos t) = \frac{s}{s^2 + 1} = F(s)$$

$$\begin{aligned} L(\cos 5t) &= \frac{1}{5} F\left(\frac{s}{5}\right) \\ &= \frac{1}{5} \left[\frac{s/5}{\left(\frac{s}{5}\right)^2 + 1} \right] \\ &= \frac{1}{5} \left[\frac{5s}{s^2 + 25} \right] \\ &= \frac{s}{s^2 + 25} \end{aligned}$$