$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds \right] dt$$

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-st}\, ds \right] dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-s} \right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t} \right] dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

$$\therefore \boxed{L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds.}$$

Problems :

(1) Find $L\left(\frac{1-\cos t}{t}\right)$

Soln:

$$L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty L\,[1-\cos t\,]\, ds$$

$$= \int_s^\infty \{ L(1) - L(\cos t) \}\, ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1} \right] ds$$

$$= \left[\log s - \frac{1}{2} \log(s^2+1) \right]_s^\infty$$

$$= \left[\log s - \log (s^2+1)^{1/2}\right]_s^{\infty}$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^{\infty}$$

$$= \left[\log \frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^{\infty}$$

$$= \log 1 - \log \left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log \left[\frac{s}{\sqrt{s^2+1}}\right]^{-1}$$

$$= \log \left[\frac{\sqrt{s^2+1}}{s}\right]$$

(2) Find $L\left(\dfrac{e^{-3t} - e^{-4t}}{t}\right)$.

Soln:

$$L(e^{-3t} - e^{-4t}) = \frac{1}{s+3} - \frac{1}{s+4}$$

$$L\left[\frac{e^{-3t} - e^{-4t}}{t}\right] = \int_s^{\infty} \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \int_s^{\infty} \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \left[\log (s+3) - \log (s+4)\right]_s^{\infty}$$

$$= \left[\log \left(\frac{s+3}{s+4}\right)\right]_s^{\infty} = \log \left(\frac{s+4}{s+3}\right)$$

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(3) Find $L\left[\frac{1-\cos at}{t}\right]$

Soln:

$$L\left[\frac{1-\cos at}{t}\right] = \int_s^\infty L(1-\cos at)\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+a^2}}\right]_s^\infty$$

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+a^2}}{s}\right).$$

(4) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

Soln:

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L[\cos at - \cos bt]\,ds$$

$$= \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

(5) Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$

Soln:

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1}$$

$$\left(\because L\int_0^t f(t)\,dt = \frac{1}{s}L[f(t)]\right)$$

$$= \left[\frac{1}{s}L(t\cos t)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{-d}{ds}L(\cos t)\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)\left((s+1)^2+1\right)^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

(6) Evaluate using Laplace transform

$$\int_0^\infty t e^{-2t}\sin 3t\,dt$$

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$$= \left[\frac{1}{s}\left(\frac{-d}{ds}L(\cos t)\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to s+1}$$

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