Initial Value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(s)$ then

$$\underset{t \to 0}{Lt} [f(t)] = \underset{s \to \infty}{Lt} [s F(s)]$$

Proof:

We know that

$$L[f'(t)] = s L[f(t)] - f(0)$$

$$= s F(s) - f(0)$$

$$\Rightarrow s F(s) = L[f'(t)] + f(0)$$

$$s F(s) = \int_0^\infty e^{-st} f'(t)\, dt + f(0).$$

Taking limit as $s \to \infty$ on both sides we get,

$$\underset{s \to \infty}{Lt} s F(s) = \underset{s \to \infty}{Lt} \left\{ \int_0^\infty e^{-st} f'(t)\, dt + f(0) \right\}$$

$$= \underset{S\to\infty}{Lt} \int_0^\infty e^{-St} f'(t)\,dt + f(0)$$

$$= \int_0^\infty \underset{S\to\infty}{Lt} e^{-St} f'(t)\,dt + f(0)$$

$$= 0 + f(0)$$

$$= \underset{t\to 0}{Lt} f(t)$$

Hence $\underset{t\to 0}{Lt} f(t) = \underset{S\to\infty}{Lt} SF(S)$.

Final value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(S)$ then

$$\underset{t\to\infty}{Lt} [f(t)] = \underset{S\to 0}{Lt} [S\,F(S)]$$

Proof:

We know that,

$$L[f'(t)] = SL[f(t)] - f(0)$$

$$= S\,F(S) - f(0)$$

$$\Rightarrow S\,F(S) = L[f'(t)] + f(0)$$

Taking Limit $S\to 0$ on both sides, we get,

$$\underset{S\to 0}{Lt} [S\,F(S)] = \underset{S\to 0}{Lt} \left\{ \int_0^\infty e^{-St} f'(t)\,dt + f(0) \right\}$$

$$= \int_0^\infty \underset{S\to 0}{Lt} e^{-St} f'(t)\,dt + f(0)$$

$$= \int_0^\infty f'(t)\,dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0) = \underset{t\to\infty}{Lt} f(t)$$

Hence $\underset{t\to\infty}{Lt} f(t) = \underset{S\to 0}{Lt} [SF(S)]$

① Verify the initial and final value theorem for

$f(t) = 1 + e^{-t}(\sin t + \cos t)$

Soln:

$$F(S) = L\left[1 + e^{-t}\sin t + e^{-t}\cos t\right]$$

$$= L(1) + L(\sin t)_{S \to S+1} + L(\cos t)_{S \to S+1}$$

$$= \frac{1}{S} + \left(\frac{1}{S^2+1}\right)_{S \to S+1} + \left(\frac{S}{S^2+1}\right)_{S \to S+1}$$

$$= \frac{1}{S} + \frac{1}{(S+1)^2+1} + \frac{S+1}{(S+1)^2+1}$$

$$= \frac{1}{S} + \frac{S+2}{S^2+2S+2}$$

$$\therefore S\,F(S) = S\left[\frac{1}{S} + \frac{S+2}{S^2+2S+2}\right]$$

Initial value theorem:

$$\underset{t \to 0}{Lt}\ f(t) = \underset{t \to 0}{Lt}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + 1 = 2$$

$$\underset{S \to \infty}{Lt}\ S\,F(S) = \underset{S \to \infty}{Lt}\ S\left[\frac{1}{S} + \frac{S+2}{S^2+2S+2}\right]$$

$$= \underset{S \to \infty}{Lt}\ 1 + \frac{S^2+2S}{S^2+2S\left(1 + \frac{2}{S^2+2S}\right)}$$

$$= 1 + 1$$

$$= 2$$

Hence $\underset{t \to 0}{Lt}\ f(t) = \underset{S \to \infty}{Lt}\ S\,F(S) = 2$

IVT is verified.

Final value theorem:

$$\underset{t \to \infty}{Lt}\ f(t) = \underset{t \to \infty}{Lt}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \lim_{s\to 0}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1$$

Hence $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 1$, FVT is Verified

Laplace Transform of Some Special functions:

Unit Step function:

The unit step function also called Heavisides unit function is defined as,

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$

This is the unit step functions at $t=a$. It can also be denoted by $H(t-a)$ or $u_a(t)$.

Result:

Laplace Transform of unit step function is $\frac{e^{-as}}{s}$ i.e., $L[u(t-a)] = \frac{e^{-as}}{s}$

Proof:

$$L[u(t-a)] = \int_0^\infty e^{-st}u(t-a)\,dt$$

$$= \int_0^a e^{-st}u(t-a)\,dt + \int_a^\infty e^{-st}u(t-a)$$

$$= 0 + \int_a^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{e^{-as}}{s} \quad (s>0)$$